PROBLEM 1

Data and notes

rectangular optical pulse:
$T_p = 100ns$ duration, $r_p = 100kHz$ pulse repetition rate;
auxiliary electrical signal available, synchronous with the optical pulse
I_e photocurrent and n_e photoelectron rate at PMT cathode

PMT:
detection efficiency $\eta = 0,1$; gain G=10^6 ; negligible excess noise F≈1;

Background emission at PMT cathode:
$n_B = 2500\, el/s$, i.e. $I_B = n_B q \approx 4 \cdot 10^{-16} A = 0,4\, fA$
Shot noise of the background current at the cathode

$$\sqrt{S_B} = \sqrt{2qI_B} = q\sqrt{2n_B} \approx 1,13 \cdot 10^{-17}\ A/\sqrt{Hz} = 11,3\, aA/\sqrt{Hz}$$

Current noise of the electronic circuitry connected to the PMT anode
$S_{iA} = 1\, pA/\sqrt{Hz}$.
It is negligible in comparison to the background current noise amplified by the PMT
$G\sqrt{S_B} \approx 11,3\, pA/\sqrt{Hz} \gg S_{iA} = 1\, pA/\sqrt{Hz}$
therefore it will not be considered

A) Measurement of amplitude of a single-pulse

A1) Optimum filter and minimum pulse amplitude

The shot noise of the background current at the cathode is

$$\sqrt{S_B} = \sqrt{2qI_B} = q\sqrt{2n_B} \approx 1,13 \cdot 10^{-17}\ A/\sqrt{Hz} = 11,3\, aA/\sqrt{Hz}$$

it is greatly amplified at the PMT anode
$G\sqrt{S_B} \approx 23\, pA/\sqrt{Hz} \gg S_{iA} = 1\, pA/\sqrt{Hz}$
and it is much higher than the electronic noise of the circuit
$\sqrt{S_{iA}} = 1\, pA/\sqrt{Hz}$
which is therefore negligible.

The PMT shot noise can be considered white, since it has autocorrelation width negligible with respect to the signal duration. Consequently, the optimum weighting function is constant over the pulse duration in any case, that is, both in case of dominant photocurrent noise and in case of dominant background noise. This optimum filtering is well approximated by a GI of duration $T_p = 100ns$ synchronized to the optical signal (a synchronized electrical signal is available to drive the GI operation).

A normalized GI has autocorrelation $k_{ww}(0) = 1/T_p$ and unilateral band-limit $f_G = 1/2T_p$, therefore in terms of currents at the photocathode we have

$$\frac{S}{N} = \frac{I_{es}}{\sqrt{\left(S_e + S_B\right)/2T_p}} = \frac{I_{es}}{\sqrt{q\left(I_{es} + I_B\right)/T_p}}$$

Minimum amplitude $I_{es,\min}$ corresponds to S/N=1

$$\frac{I_{es,\min}}{\sqrt{q\left(I_{es,\min} + I_B\right)/T_p}} = 1$$

If $I_B \ll I_{es,\min}$ the background is negligible and we have

$$\frac{I_{es,\min}}{\sqrt{qI_{es,\min}/T_p}} \approx 1$$

which leads to

$$I_{es,\min} = \frac{q}{T_p} = 1,6 \cdot 10^{-12} A = 1,6\, pA$$

We can verify that

$$I_{e,\min 1} = 1,6\, pA \gg I_B = 0,4\, fA$$

that is, in this case the approximation is valid, the photocurrent noise is dominant and we have correctly computed the minimum measurable pulse amplitude.

A2) Minimum pulse in terms of number of detected photons

The minimum signal corresponds to just one photon detected per pulse, hence to a photoelectron rate

$$n_{es,\min} = \frac{I_{es,\min}}{q} = \frac{1}{T_p} = 10^7\, el/s$$

(which is indeed much higher than the background emission rate $n_B = 2500\, el/s$)

The minimum measurable amplitude of the optical pulse in terms of photon rate is

$$n_{ps,\min} = \frac{n_{es,\min}}{\eta} = 10^8\, ph/s$$

B) Measurement of pulse amplitude averaged over a moderate number N_{m1}=100 of pulses

B1) Minimum pulse amplitude

The average over 100 measured pulses with pulse repetition rate $r_p = 100kHz$ is still quite fast, only 1ms is necessary for collecting 100 pulses at $r_p = 100kHz$.

The measurements of the various pulses are uncorrelated, hence mean values and mean square deviations of the single pulses are added in the averaged measurement.

$$\frac{S}{N} = \frac{N_{m1} I_{e,m1}}{\sqrt{N_{m1} q\left(I_{e,m1} + I_B\right)/T_p}} = \frac{I_{e,m1}}{\sqrt{q\left(I_{e,m1} + I_B\right)/N_{m1}T_p}} =$$

Hence

$$\frac{I_{e,m1,\min}}{\sqrt{q\left(I_{e,m1,\min}+I_B\right)/N_{m1}T_p}}=1$$

Assuming $I_B \ll I_{e,m1,\min}$ the background is negligible and

$$\frac{I_{e,m1,\min}}{\sqrt{qI_{e,m1,\min}/N_{m1}T_p}}=1$$

which leads to

$$I_{e,m1,\min}=\frac{q}{T_pN_{m1}}=1,6\cdot 10^{-14}A=16fA$$

We can verify that

$I_{e,m1,\min}=16fA \gg I_B=0,4fA$

that is, the approximation is valid also in this case, the photocurrent noise is still dominant and we have correctly computed the minimum measurable pulse amplitude in a measure averaged over $N_{m1}=100$ pulses.

The minimum signal corresponds to a photoelectron rate

$$n_{e,m1,\min}=\frac{I_{e,m1,\min}}{q}=\frac{1}{N_{m1}T_p}=10^5 el/s$$

which corresponds to a minimum amplitude of the optical pulse in terms of photon rate is

$$n_{p,m1,\min}=\frac{n_{e,m1,\min}}{\eta}=10^6 ph/s$$

B2) Dependence of minimum pulse amplitude on the (moderate) number of pulses averaged

In a measure averaged over a moderate number (around 100) of pulses the minimum measurable amplitude becomes progressively lower as N_{m1} increases, but it is still high enough to keep the photocurrent noise contribution dominant with respect to the background noise. Therefore, the minimum measured amplitude decreases as the reciprocal of the number of pulses N_{m1}

$$I_{e,m1,\min}=\frac{q}{T_pN_{m1}}=\frac{I_{es,\min}}{N_{m1}}=\frac{I_{es,\min}}{100}$$

C) Measurement of pulse amplitude averaged over a high number $N_{m2}=10^6$ of pulses

C1) Minimum pulse amplitude

The measurement averaged over 10^6 pulses takes 10s (with pulse repetition rate $r_p=100kHz$).
The measurements of the various pulses are uncorrelated, hence mean values and mean square deviations of the single pulses are added in the averaged measurement.

$$\frac{S}{N}=\frac{N_{m2}I_e}{\sqrt{N_{m2}q\left(I_e+I_B\right)/T_p}}=\frac{I_e}{\sqrt{q\left(I_e+I_B\right)/N_{m2}T_p}}=$$

Hence

$$\frac{I_{e,m2,\min}}{\sqrt{q\left(I_{e,m2,\min}+I_B\right)/N_{m2}T_p}}=1$$

Let us assume first that still it is $I_B \ll I_{e,m2,\min}$, i.e. that the background noise be still negligible with respect to the photocurrent noise. We get

$$\frac{I_{e,m2,\min}}{\sqrt{qI_{e,m2,\min}/N_{m2}T_p}}=1$$

However, this leads to

$$I_{e,m2,\min}=\frac{q}{N_{m2}T_p}=\frac{I_{e,\min 1}}{N_{m2}}=1,6\cdot 10^{-18}\,A=1,6\;aA$$

which shows that the assumption is unfounded, the approximation of negligible background noise is not valid in this case

$I_B = 0,4\,fA \gg I_{e,m2,\min} = 1,6aA$

Let us see then if we can get a correct result with the opposite approximation, that is, by considering negligible the photocurrent noise with respect to background noise. We get

$$\frac{I_{e,m2,\min}}{\sqrt{qI_B/N_{m2}T_p}}=1$$

Denoting by $N_{Bp}=n_BT_p=10^{-3}$ the mean number of background electrons emitted per pulse, we get

$$I_{e,m2,\min}\approx\sqrt{\frac{qI_B}{N_{m2}T_p}}=\frac{q}{T_p}\sqrt{\frac{n_BT_p}{N_{m2}}}=\frac{q}{T_p}\sqrt{\frac{N_{Bp}}{N_{m2}}}=25aA$$

in this case it is confirmed that the approximation of negligible photocurrent noise is valid

$I_{e,m2,\min}=25aA \ll I_B = 400aA$

C2) Dependence of the minimum pulse amplitude on the high number of pulses averaged

In a measure averaged over a high number (around 10^6) of pulses N_{m2} the minimum measured pulse amplitude becomes so low that the background noise contribution becomes dominant. This causes the minimum measured amplitude to <u>decrease as the reciprocal of the square root of the number of pulses $\sqrt{N_{m2}}$</u>

With respect to the measurement on a single pulse, the measurement averaged over a high number of pulses gives minimum measurable amplitude improved by the factor $\sqrt{N_B/N_{m2}}$

$$I_{e,m2,\min}=\frac{q}{T_p}\sqrt{\frac{N_B}{N_{m2}}}=I_{es,\min}\sqrt{\frac{N_B}{N_{m2}}}$$

In our case with $N_B = 10^{-3}$ and $N_{m2} = 10^6$ we get

$$I_{e,m2,\min} = \frac{I_{es,\min}}{\sqrt{10^9}} \approx \frac{I_{es,\min}}{3,2 \cdot 10^4}$$

D) Baseline subtraction

In measurement of single pulses (see Section A) the baseline current I_B is not significant, hence it is not necessary to subtract it from the measured current. In fact, I_B is negligible even in comparison to the minimum measurable pulse current.

In measurements averaged over 100 pulses (see Section B) it is still not necessary to subtract the baseline I_B from the measured current, since it is much smaller than the minimum measurable pulse current.

In measurements averaged over 10^6 pulses (see Section C) the measurable pulse amplitude becomes so low that the baseline becomes significant and must be separately measured and subtracted.

The baseline can be measured by means of a GI equal to that employed for measuring the signal pulse and open in a time interval with equal duration, but after the optical pulse (or before it).

The uncorrelated noise brought by the subtracted baseline doubles the background noise in the pulse measurement. Therefore, we have

$$\frac{S}{N} = \frac{N_{m2} I_e}{\sqrt{N_{m2} q \left(I_e + 2I_B\right)/T_p}} = \frac{I_e}{\sqrt{q\left(I_e + 2I_B\right)/N_{m2}T_p}}$$

Hence

$$\frac{I_{e,m2,\min}}{\sqrt{q\left(I_{e,m2,\min} + 2I_B\right)/N_{m2}T_p}} = 1$$

With the high number of pulses $N_{m2} = 10^6$ the photocurrent noise is negligible with respect to background noise (as already seen without baseline subtraction) and we get now

$$\frac{I_{e,m2,\min}}{\sqrt{2qI_B/N_{m2}T_p}} = 1$$

Denoting by $N_{Bp} = n_B T_p = 10^{-3}$ the mean number of background electrons emitted per pulse, we get

$$I_{e,m2,\min} \approx \sqrt{\frac{2qI_B}{N_{m2}T_p}} = \frac{q}{T_p}\sqrt{\frac{2n_B T_p}{N_{m2}}} = \frac{q}{T_p}\sqrt{\frac{2N_{Bp}}{N_{m2}}} = 35aA$$

The minimum amplitude is increased by the factor $\sqrt{2}$ because the dominant noise of the background is doubled by the baseline subtraction.