PROBLEM 1

Data and notes

Signal: rectangular pulse with

T_P = 500 μs duration

V_P amplitude, to be measured

Noise:

$\sqrt{S_{V,u}} = 40 nV/\sqrt{Hz}$ (unilateral) white with wide band

Noise band-limit $f_n = 1/4T_n = 2,5MHz$ f_n= 2,5MHz

Noise correlation time $T_n = 1/4 f_n = 100ns$

In part (D) consider that the band-limit and correlation time are produced by a single-pole low-pass filtering with time constant T_n and pole frequency $f_p = 1/2\pi T_n$.

Sampling

$f_S = 1/T_S$ sampling frequency, selectable

$T_S = 1/f_S$ sampling interval

(A) Optimum filtering with continuous weighting

With white noise the optimum weighting function is equal to the signal waveform. In this case it is well approximated by a Gated Integrator (GI) normalized to unit gain

$$w_{op}(t) = \frac{1}{T_P} \quad \text{in } 0<t<T_P$$

At the GI output:

signal $s_y = V_P$ noise $\sqrt{\overline{n^2_{y,op}}} = \frac{\sqrt{S_{V,u}}}{\sqrt{2T_P}}$ $\left(\frac{S}{N}\right)_{op} = \frac{V_P}{\sqrt{S_{V,u}}}\sqrt{2T_P}$

minimum measurable $V_{P\min,op} = \sqrt{\overline{n^2_{y,op}}} = \frac{\sqrt{S_{V,u}}}{\sqrt{2T_P}} \approx 1,3\mu V$

Without filtering:

signal $s_y = V_P$ noise $\sqrt{\overline{n^2_x}} = \frac{\sqrt{S_{V,u}}}{\sqrt{4T_n}}$ $\left(\frac{S}{N}\right)_x = \frac{V_P}{\sqrt{S_{V,u}}}\sqrt{4T_n}$

Thus we have

$\overline{n^2_{y,op}} = \overline{n^2_x}\frac{2T_n}{T_P}$ and the factor of improvement is

$$\sqrt{\frac{\overline{n^2_x}}{\overline{n^2_{y,op}}}} = \sqrt{\frac{T_P}{2T_n}} = \sqrt{2500} \approx 50$$

This high factor is intuitively explained by the strong reduction of bandwidth brought by the filter: the factor is indeed set by the ratio of the bandwidth $1/4T_n$=2,5MHz of the input noise to the bandwidth $1/2T_p$=1kHz of the noise filtering by the GI.

(B) <u>Filtering by averaging of discrete samples</u>

The signal duration T_P is divided in N time intervals of duration T_s

$$N = \frac{T_P}{T_s}$$

The signal amplitude is sampled at the center of each interval T_s and the samples are summed weighted by a weight 1/N, in order to have filtering normalized to unit gain. The output signal is

$$s_y = N\frac{V_P}{N} = V_P$$

With uncorrelated noise samples

$$\sqrt{\overline{n^2_{y,s}}} = \sqrt{N\frac{\overline{n^2_x}}{N^2}} = \frac{\sqrt{\overline{n^2_x}}}{\sqrt{N}} = \frac{1}{\sqrt{N}}\frac{\sqrt{S_{V,u}}}{\sqrt{4T_h}}$$

$$\left(\frac{S}{N}\right)_s = V_P\frac{\sqrt{4T_n}}{\sqrt{S_{V,u}}}\sqrt{N}$$

The factor of improvement is $\sqrt{N}$.

(C) <u>Approximate evaluation of the dependance of filtering results on the sampling frequency</u>

The dependence of the result on the sampling frequency $f_s = 1/T_s$ is explicited by substituting in the equation $N=f_sT_P$

$$\left(\frac{S}{N}\right)_s = V_P\frac{\sqrt{4T_n}}{\sqrt{S_{V,u}}}\sqrt{f_sT_P} = V_P\frac{\sqrt{4T_n}}{\sqrt{S_{V,u}}}\sqrt{\frac{T_P}{T_s}}$$

However, this equation is valid only as long as the noise samples are uncorrelated.

In general terms, the output noise has to be computed (in time or in frequency domain) by taking into account the features of the noise and of the filter weighting in the equation

$$\overline{n^2_y} = \int_{-\infty}^{\infty} k_{ww}(\tau)R_{nn}(\tau)d\tau = \int_{-\infty}^{\infty}\left|W(f)\right|^2 S_{V,b}(f)df = \int_0^{\infty}\left|W(f)\right|^2 S_{V,u}(f)df$$

Filtering by sample averaging has a weighting function $w(\tau)$ which is a set of δ-functions with weight 1/N spaced by T_s over the signal duration T_P . The corresponding autocorrelation function $k_{ww}(\tau)$ is a set of δ-functions with weight decreasing from 1/N at τ=0 to zero at the edge τ=T_P. This weight can thus be considered with good approximation a constant 1/N over all the small time range (about $5T_n$, hence <<T_p) covered by the noise autocorrelation $R_{nn}(\tau)$.

In frequency, the noise-weighting function $\left|W(f)\right|^2$ is the transform of $k_{ww}(\tau)$. It is a set of narrow-width pulse-function in frequency spaced by $1/T_s$ (they are actually $(\text{sync})^2$ functions with unit amplitude and width $1/2T_p$).

In the approximate treatment, the wide-band noise has

- spectrum of rectangular shape with constant bilateral density $S_{V,b}$ up to frequency $f_n = 1/4T_n = 2,5MHz$
- autocorrelation function of triangular shape with maximum value $\overline{n_x^2}$ and base width $4T_n$.

With this approximate representation of the noise, we see that

1. The equations derived for the S/N and for the output noise of the sample averaging is valid only as long as $T_S \geq 2T_n$, that is, as long as only the central δ(τ) of the $k_{ww}(\tau)$ overlaps $R_{nn}(\tau)$. Equivalently, it is valid as long $f_S \leq 1/2T_n$, which implies that more than one pulse-function of $\left|W(f)\right|^2$ overlaps the $S_{V,b}$ rectangle. Within this range $f_S \leq 1/2T_n$ the S/N steadily increases as $\sqrt{f_S}$.
2. With $f_S = 1/2T_n$ (i.e, $T_S = 2T_n$) the result obtained is equal to that of the optimum filter.
3. With $f_S > 1/2T_n$ (i.e, $T_S < 2T_n$) the result remains constant at the level achieved with $f_{S,m} = 1/2T_n$. In the analysis based on the noise spectrum this conclusion is quite evident: at any f_s in this range only the central pulse-function of $\left|W(f)\right|^2$ overlaps the $S_{V,b}$ rectangle. In the analysis with the autocorrelation function the conclusion is confirmed simply by computing the noise with the time-domain equation

In summary: the S/N increases as $\sqrt{f_S}$ as long as $f_S \leq 1/2T_n$; at $f_S = 1/2T_n$ the S/N reaches the optimum value; for $f_S > 1/2T_n$ the S/N stays constant at the optimum level. Therefore, the conclusion is evident: the sampling advisable frequency is just $f_{S,m} = 1/2T_n$, since increasing it further does not bring any further improvement.

(D) <u>Evaluation of the actual dependance of the filtering result on the sampling frequency</u>

Since we know that the noise band-limit is produced by a single-pole LPF filter with time constant T_n and pole frequency $f_p = 1/2\pi T_n$, we know in detail its autocorrelation function and spectrum

$$R_{nn}(\tau) = \overline{n_x^2} \exp\left(-\left|\frac{\tau}{T_n}\right|\right) \qquad \text{and} \qquad S_{V,u}(\omega) = \frac{S_{V,u}(0)}{1+\omega^2 T_n^2}$$

The actual output noise can be thus be accurately computed by the time-domain equation

$$\overline{n_y^2} = \int_{-\infty}^{\infty} k_{ww}(\tau) R_{nn}(\tau)\, d\tau = \int_{-\infty}^{\infty} k_{ww}(\tau)\, \overline{n_x^2} \exp\left(-\left|\frac{\tau}{T_n}\right|\right) d\tau$$

or by the frequency-domain equation.

$$\overline{n_y^2} = \int_0^{\infty} \left|W(f)\right|^2 S_{V,u}(f)\, df = \int_0^{\infty} \left|W(f)\right|^2 \frac{S_{V,u}(0)}{1+(2\pi f)^2 T_n^2} df$$

Observing the differences between the approximate and the true $k_{ww}(\tau)$ and $\left|W(f)\right|^2$ functions, the main differences between the approximate and the true dependence on the sampling frequency are fairly clear

a) The improvement of the result with increasing f_s is more gradual and smooth than that computed in the approximate treatment

b) at frequencies f_s near to $f_{S,m} = 1/2T_n$ the true noise is somewhat higher than the approximate value, since the true autocorrelation function of the noise is higher than the approximate one

c) There is not a finite frequency where the optimum result is reached. The optimum is reached only asymptotically as the sampling frequency is increased. As intuitive, discrete filtering with very closely spaced samples is equivalent to continuous filtering.

The conclusion that $f_{S,m} = 1/2T_n$ leads to the optimum result must be revised. Increasing further f_s brings further improvement, but the rate of increase gests progressively slower. The advisable selection of f_s may thus depend also on auxiliary practical considerations.

Appendix: computation of output noise with noise spectrum limited by single-pole filtering

The computation of the true noise in the frequency-domain is not difficult. In the equation

$$\overline{n_y^2} = \int_{-\infty}^{\infty} k_{ww}(\tau) R_{nn}(\tau) d\tau = \int_{-\infty}^{\infty} k_{ww}(\tau) \overline{n_x^2} \exp\left(-\left| \frac{\tau}{T_n} \right| \right) d\tau$$

the δ-functions in k_{ww} are at: τ=0 , $\tau = \pm T_S$ $\tau = \pm 2T_S$ $\tau = \pm 3T_S$ ....

their weight is with good approximation a constant 1/N over the interval occupied by R_{nn} (the relative decrease is $5T_n / T_p \approx 10^{-3}$).

Denoting by α the ratio of the sampling interval T_S to the autocorrelation time T_n

$$T_S = \alpha T_n$$

we get

$$\overline{n_y^2} = \int_{-\infty}^{\infty} k_{ww}(\tau) R_{nn}(\tau) d\tau = \frac{\overline{n_x^2}}{N} + 2\frac{\overline{n_x^2}}{N} e^{-\alpha} + 2\frac{\overline{n_x^2}}{N} e^{-2\alpha} + 2\frac{\overline{n_x^2}}{N} e^{-3\alpha} .....$$

That is

$$\overline{n_y^2} = \frac{\overline{n_x^2}}{N}\left[1 + 2e^{-\alpha} + 2e^{-2\alpha} + ...\right] = \frac{\overline{n_x^2}}{N}\left\{2\left[1 + e^{-\alpha} + e^{-2\alpha} + ...\right] - 1\right\} = \frac{\overline{n_x^2}}{N}\left\{\frac{2}{1-e^{-\alpha}} - 1\right\} = \frac{\overline{n_x^2}}{N} \cdot \frac{1+e^{-\alpha}}{1-e^{-\alpha}}$$

Since it is $N = \frac{T_p}{T_S}$ we get

$$\overline{n_y^2} = \frac{\overline{n_x^2}}{N} \cdot \frac{1+e^{-\alpha}}{1-e^{-\alpha}} = \overline{n_x^2} \frac{T_S}{T_p} \cdot \frac{1+e^{-\alpha}}{1-e^{-\alpha}} = \overline{n_x^2} \frac{\alpha T_n}{T_p} \cdot \frac{1+e^{-\alpha}}{1-e^{-\alpha}}$$

Recalling that the optimum is

$$\overline{n_{y,op}^2} = \overline{n_x^2} \frac{2T_n}{T_P}$$

we can directly compare the computed result with that of the optimum filtering

$$\overline{n_y^2} = \frac{2T_n}{T_p} \cdot \frac{\alpha}{2} \frac{1+e^{-\alpha}}{1-e^{-\alpha}} = \overline{n_{y,op}^2} \cdot \frac{\alpha}{2} \cdot \frac{1+e^{-\alpha}}{1-e^{-\alpha}}$$

We can here verify quantitatively some qualitative conclusion drawn in (C).

For instance, sampling at $f_{S,m} = 1/2T_n$ (i.e. with α=2) does not give the optimum result, it is about 30% worse than it

$$\overline{n_y^2} = \overline{n_{y,op}^2} \cdot \frac{1+e^{-2}}{1-e^{-2}} \approx 1,31 \cdot \overline{n_{y,op}^2}$$

We can also verify that the optimum is reached asymptotically: for α<<1 we get indeed

$$\frac{\alpha}{2} \cdot \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \to \frac{\alpha}{2} \cdot \frac{1+1+\alpha}{\alpha} \to 1$$