PROBLEM 1

SPECIFIED DATA

Preamplifier

Noise referred to the input $\sqrt{S_{bu}} = \sqrt{2S_{bb}} = 40\, nV/\sqrt{Hz}$ (unilateral density)

Band limited by a pole at $f_p = 100MHz$

Rectangular pulse

Duration T_r=4 μs Amplitude A_r

Exponential pulse

$x_e(t) = A_e e^{-t/T_e}$ with time constant $T_e = 2\mu s$ amplitude A_e

(A) Comparing the measurable pulse amplitude without filtering and with optimum filtering

Without filtering: the noise is collected over the entire preamplifier band

$$\sqrt{\overline{n_u^2}} = \sqrt{S_b f_{pa}} = \sqrt{S_b \frac{\pi}{2} f_p} = \quad 500\mu V$$

The minimum measurable amplitude is equal for both pulses

$$A_{r,\min} = A_{e,\min} = \sqrt{\overline{n_u^2}} = \sqrt{S_b}\sqrt{\frac{\pi}{2} f_p} \approx 500\mu V$$

Optimum filtering: since the noise is white over the entire band, the weighting function of the optimum filter is equal to the waveform of the signal pulse.

1) Rectangular pulse

Pulse signal $x_r = A_r$ for $0<t< T_r$

Optimum weighting function $w_r = \frac{1}{A_r}$ for $0<t< T_r$

Output signal $y_{ro} = \int_0^{T_r} x_r w_r dt = A_r$

Output noise $\overline{n_{ro}^2} = S_{bb} k_{ww}(0) = S_{bb} \int_0^{\infty} w_r^2 dt = \frac{S_{bb}}{T_r}$

Output S/N $\left(\frac{S}{N}\right)_{ro} = \frac{A_r}{\sqrt{\frac{S_{bb}}{T_r}}} = \frac{A_r}{\sqrt{\frac{S_{bu}}{2T_r}}}$

Minimum pulse amplitude $A_{r,\min} = \sqrt{\dfrac{S_{bu}}{2T_r}} = 14{,}2\ \mu\text{V}$

2) Exponential pulse

Pulse signal $x_e = A_e e^{-t/T_e}$ for $t>0$

Optimum weighting function $w_e = \dfrac{1}{T_e} e^{-t/T_e}$

Output signal $y_{eo} = \int_0^{T_r} x_e w_e dt = A_e \int_0^{\infty} e^{-2t/T_e} \dfrac{1}{T_e} dt = \dfrac{A_e}{2}$

Output noise $\overline{n_{eo}^2} = S_{bb} k_{ww}(0) = S_{bb} \int_0^{\infty} w_e^2 dt = S_{bb} \int_0^{\infty} e^{-2t/T_e} \dfrac{1}{T_e^2} dt = \dfrac{S_{bb}}{2T_e}$

Output S/N $\left(\dfrac{S}{N}\right)_{eo} = \dfrac{A_e}{\sqrt{\dfrac{2S_{bb}}{T_e}}}$

Minimum pulse amplitude $A_{eo,\min} = \sqrt{\dfrac{2S_{bb}}{T_e}} = \sqrt{\dfrac{S_{bu}}{T_e}} = 28{,}3\mu\text{V}$

3) Comparation of results with rectangular and exponential pulses having $T_r = 2T_e$

Let us consider input pulses with $T_r = 2T_e$ and equal amplitude $A_r = A_e$, each one filtered by its optimum filter. We note that

- the output amplitude of the optimally filtered rectangular pulse is higher

$$y_{ro} = A_r \qquad \text{and} \qquad y_{eo} = \frac{A_e}{2} = \frac{y_{ro}}{2}$$

- the output noise power is equal in the two cases,

$$\overline{n_{ro}^2} = \frac{S_{bb}}{T_r} = \frac{S_{bb}}{2T_e} = \overline{n_{eo}^2}$$

- therefore, the minimum measurable pulse is smaller for the rectangular pulse

$$A_{r,\min} = \sqrt{\frac{S_{bb}}{T_r}} = \qquad\qquad A_{e,\min} = \sqrt{\frac{2S_{bb}}{T_e}} = 2\sqrt{\frac{S_{bb}}{T_e}} = 2A_{r,\min}$$

The result obtained with the rectangular pulse is better and this can be intuitively explained by comparing the features of the two pulses. For instance, the greater energy of the input rectangular pulse points out that the optimized S/N of this pulse is greater.

(B) measurable amplitude with practically implementable continuous analog filters that approximate the optimum filter

1) Rectangular Pulse

The optimal filtering can be very well approximated with a mobile-mean integrator. This circuit has indeed a rectangular weighting function with duration T_r set by a delay line, which can be fairly well implemented in practice for delays with moderate values, as in the case here considered. In this case, we have a practically implementable circuit that gives an excellent approximation of the optimum filter. Therefore, the computations and conclusions seen in Sec. A1 are valid also for this case

2) Exponential pulse

We may try to approximate the optimum filter with an RC integrator circuit, which has an exponentially decaying delta-response *h(t)*

$$h(t) = \frac{1}{RC} e^{-t/RC}$$

However, the filter weighting function is the delta-response reversed in time. Therefore, the weighting function of this approximate-optimum filter is reversed in time with respect to that of the true optimum filter. The approximation is moderate, but it is based on a simple and effective filter and is worth evaluating it. A comparison of the plot of *h(t)* to that of the optimum filter weighting function

$$w_e = \frac{1}{T_e} e^{-t/T_e}$$

suggests to select a filter time constant RC equal to the signal time constant T_e . This selection is supported by the fact that this approximate-optimum filter gives the same output noise as the true optimum filter. In fact, by reversing in time a function, its autocorrelation does not change. Therefore, the output noise is the same computed in Sec. A 2.

The output signal waveform $y_e(t)$ can be obtained by convolution of input pulse and delta-response

$$y_e(t) = x_e(t) * h(t) = A_e \int_0^t e^{-\alpha/T_e} e^{-(t-\alpha)/T_e} \frac{dt}{T_e} = A_e \frac{t}{T_e} e^{-t/T_e}$$

The maximum output amplitude is

$$y_{em} = \frac{A_e}{e}$$

and by recalling from Sec. A 2 the output noise

$$\overline{n_e^2} = \overline{n_{eo}^2} = S_{bb} k_{ww}(0) = S_{bb} \int_0^\infty w_e^2 dt = S_{bb} \int_0^\infty e^{-2t/T_e} \frac{1}{T_e^2} dt = \frac{S_{bb}}{2T_e}$$

We get S/N
$$\left(\frac{S}{N}\right)_e = \frac{A_e}{e}\frac{1}{\sqrt{\dfrac{S_{bb}}{2T_e}}}$$

And therefore the minimum pulse amplitude

$$A_{e,\min} = e\sqrt{\frac{S_{bb}}{2T_e}} = \frac{e}{2}\sqrt{\frac{2S_{bb}}{T_e}} = \frac{e}{2}A_{eo,\min} \approx 1{,}36 A_{eo,\min} = 38{,}5\ \mu\mathrm{V}$$

is 36% higher than the true optimum

(C) <u>measurable amplitude with discrete analog filters that approximate the optimum filter</u>

1) Rectangular pulse

The optimum weighting function can be well approximated by a discrete integrator (DI) with samples taken over T_r spaced by an interval Δt_s and weighted with constant weight $1/T_r$. This is evident from the plot of the function and is easily checked by computation

Signal output
$$y_r = \sum_j w_j x_j \Delta t_s = \sum_j A_r \frac{\Delta t_s}{T_r} = A_r$$

Noise output
$$\overline{n_r^2} = S_{bb}k_{ww}(0) = S_{bb}\sum_j w_j^2 \Delta t_s = S_{bb}\sum_j \frac{1}{T_r^2}\Delta t_s = S_{bb}\frac{1}{T_r} = S_{bu}\frac{1}{2T_r} \approx$$

Output S/N
$$\left(\frac{S}{N}\right)_r = \frac{A_r}{\sqrt{\dfrac{S_{bb}}{T_r}}}$$

Minimum pulse amplitude
$$A_{r,\min} = \sqrt{\frac{S_{bb}}{T_r}} = \sqrt{\frac{S_{bu}}{2T_r}} \approx 14{,}2\ \mu\mathrm{V}$$

We conclude that in the case of rectangular pulse it is easy to obtain a good approximation of the optimum filtering both with continuous filtering and with discrete filtering.

2) Exponential pulse

Discrete-time sampling followed by digital conversion and elaboration of samples gives great flexibility in the selection of the filter weighting. It is thus possible to obtain a good approximation of the optimum weighting also in case of exponential pulse, by selecting a distribution of the discrete weights that closely approximates the continuous exponential weighting. In order to achieve this, it is evident from the plots and it can be confirmed by computation that the time spacing between samples must be small with respect to the time constant T_e of the pulse.

With sampling interval Δt and weighting $w_j = \dfrac{1}{T_e} e^{-j\Delta t/T_e}$

the sampled signal is $x_j = A_e e^{-j\Delta t/T_e}$

The output signal is $y_e = \sum_j w_j x_j \Delta t = A_e \sum_j e^{-j2\Delta t/T_e} \dfrac{\Delta t}{T_e}$

by denoting $\alpha = \dfrac{\Delta t}{T_e}$ (with α<<1)

we get $y_e = A_e \alpha \sum_j e^{-j2\alpha} = A_e \dfrac{\alpha}{1-e^{-2\alpha}}$

with sufficienly small α, i.e. with α<<1 we get

$$y_e = A_e \frac{\alpha}{1-e^{-2\alpha}} \approx A_e \frac{\alpha}{1-1+2\alpha} = \frac{A_e}{2}$$

The output noise is

$$\overline{n_e^2} = S_{bb} k_{ww}(0) = S_{bb} \sum_j w_j^2 \Delta t = S_{bb} \frac{1}{T_r^2} \Delta t \sum_j e^{-j2\alpha} = S_{bb} \frac{\alpha}{T_r} \frac{1}{1-e^{-2\alpha}}$$

with sufficienly small α, i.e. with α<<1 we get

$$\overline{n_e^2} = S_{bb} \frac{\alpha}{T_r} \frac{1}{1-e^{-2\alpha}} \approx S_{bb} \frac{\alpha}{T_r} \frac{\alpha}{1-1+2\alpha} = S_{bb} \frac{1}{2T_e}$$

In conclusion, with this approximation we get S/N and minimal measurable amplitude practically equal to the optimum

Output S/N $\left(\dfrac{S}{N}\right)_e = \dfrac{A_e}{\sqrt{\dfrac{2S_{bb}}{T_e}}}$

Minimum pulse amplitude $A_{e,\min} = \sqrt{\dfrac{2S_{bb}}{T_e}} = 28{,}3\mu V$