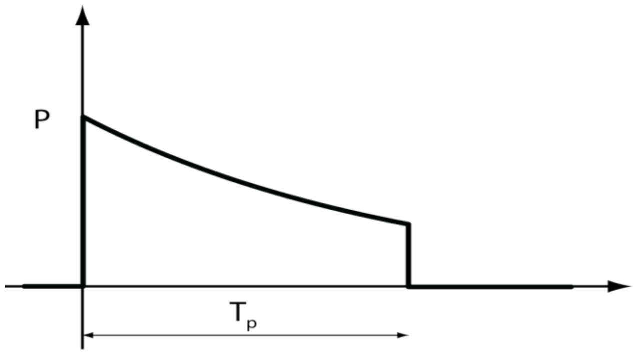
<u>PROBLEM 1</u>

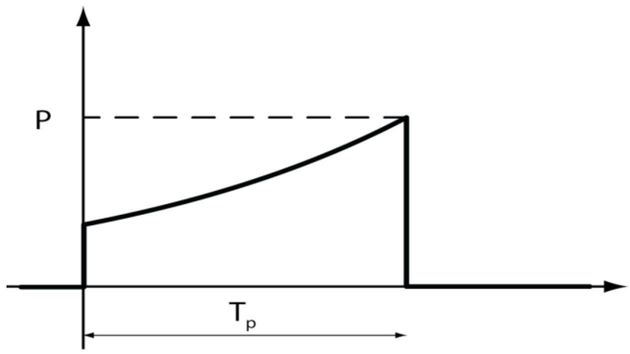
Data

$x(t) = P\exp(-t/T_D)$ in $0 < t < T_P$



Exponential Decaying Pulse

$x(t) = P\exp\left[(t - T_P)/T_D\right]$ in $0 < t < T_P$



Exponential Rising Pulse

Pulse duration T_P = 10 μs Time constant T_D = 10 μs Peak amplitude P	White Noise spectral density (unilateral) $S_{Vu}^{1/2} = 14\,nV/\sqrt{Hz}$ SV)1/2 = 14 nV/Hz1/2

A) Exponential decaying pulse

Signal pulse waveform $x(t) = P\exp(-t/T_D)$ for 0<t<T_P

Signal area $A = PT_D\left[1 - \exp(-T_P/T_D)\right]$

Normalized signal $b(t) = \dfrac{x(t)}{A} = \dfrac{\exp(-t/T_D)}{T_D\left[1 - \exp(-T_P/T_D)\right]}$

A1) Optimum filtering

In cases with white noise, the optimum weighting filter is the matched filter, which has waveform equal to the signal pulse

Optimum weighting $w(t) = b(t)$

Weighting autocorrelation $k_{ww}0 = \int_0^{T_P} w^2(t)\,dt = \dfrac{1}{2T_D} \cdot \dfrac{1 - \exp(-2T_P/T_D)}{\left[1 - \exp(-T_P/T_D)\right]^2}$

Output Noise Power $\overline{n_U^2} = S_{V,b} \cdot k_{ww}(0) = \dfrac{S_{V,b}}{2T_D} \cdot \dfrac{1 - \exp(-2T_P/T_D)}{\left[1 - \exp(-T_P/T_D)\right]^2}$

Output signal $u_s = \int_0^{T_P} w(t)\,x(t)\,dt = \dfrac{P}{2} \cdot \dfrac{1 - \exp(-2T_P/T_D)}{\left[1 - \exp(-T_P/T_D)\right]}$

$$\left(\frac{S}{N}\right)_{u,o} = \frac{u_s}{\sqrt{\overline{n_u^2}}} = \frac{P}{\sqrt{S_{v,b}}}\sqrt{\frac{T_D}{2}}\sqrt{1 - \exp(-2T_P/T_D)}$$

Minimum signal $P_{\min} = \sqrt{S_{v,b}}\sqrt{\dfrac{2}{T_D}}\dfrac{1}{\sqrt{1 - \exp(-2T_P/T_D)}}$

which with $T_D = T_P$

leads to $P_{\min} = \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\sqrt{\frac{e^2}{e^2-1}} = 1,07\sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}$

In our case with

$\sqrt{S_{v,b}} = \sqrt{\frac{S_{v,u}}{2}} = 10\,nV/\sqrt{Hz}$ and $T_D = 10\mu s$

we get $P_{\min,o} = 1,07 \cdot 4,47 = 4,8\mu V$

A2) Filtering with simple RC lowpass filter

Filtering with a simple RC lowpass filter, we have

Weighting function

$$w(t) = \frac{1}{T_F}\exp\left[\frac{t-T_P}{T_F}\right] \qquad \text{for } t<T_P$$

We note that the weighting function is more similar to the optimum weighting function with the advised selection $T_F = T_D$, which we adopt.

Weighting autocorrelation $k_{ww}(0) = \frac{1}{2T_F}$ which with $T_F = T_D$ reduces to $k_{ww}(0) = \frac{1}{2T_D}$

Output Noise Power $\overline{n_U^2} = S_{V,b} \cdot k_{ww}(0) = \frac{S_{V,b}}{2T_F} = \frac{S_{V,b}}{2T_D}$

From start to T_P, the signal at the filter output is the same as the well known case of input exponential signal with equal time constant T_D given to the RC filter. The output thus reaches the maximum amplitude at time T_P

$$u_s = \frac{T_P}{T_D}\exp\left(-\frac{T_P}{T_D}\right)$$

with $T_P = T_D$ it is $u_s = \frac{P}{e}$

$$\left(\frac{S}{N}\right) = \frac{P}{e}\frac{\sqrt{2T_D}}{\sqrt{S_{v,b}}} = \frac{P}{\sqrt{S_{v,b}}}\sqrt{\frac{T_D}{2}}\frac{2}{e}$$

Minimum signal $P_{\min} = \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\frac{e}{2} = 1,36\sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}$

That is, with $\sqrt{S_{v,b}} = 10\,nV/\sqrt{Hz}$ and $T_D = 10\mu s$

$$P_{\min} = 1,36 \cdot 4,47\mu V = 6,1\mu V$$

With respect to the optimum filtering, the result is worse by a remarkable factor

$$\frac{1,36}{1,07} = 1,27$$

This is due mainly to the remarkable difference between the RC weighting and the optimum weighting in the filtering of the signal.

A3) Filtering with a Gated Integrator (GI)

Denoting by T_G the gating interval

Weighting function $\quad w = \dfrac{1}{T_G} \quad$ in $\quad 0<t<T_G$

In order to approximate the optimum weighting we select

$$T_G = T_P = T_D$$

So that $\quad w = \dfrac{1}{T_D} \quad$ in $\quad 0<t<T_P$

Weighting autocorrelation $\quad k_{ww}(0) = \dfrac{1}{2T_D}$

Output signal $\quad u_s = \int_0^{T_P} w(t)\, x(t)\, dt = P\left[1-\exp\left(-T_P/T_D\right)\right]$

$$\left(\frac{S}{N}\right)_u = \frac{u_s}{\sqrt{\overline{n_u^2}}} = \frac{P}{\sqrt{S_{v,b}}} \frac{1-\exp\left(-T_P/T_D\right)}{\dfrac{1}{\sqrt{T_D}}} = \frac{P}{\sqrt{S_{v,b}}}\sqrt{\frac{T_D}{2}}\sqrt{2}\left[1-\exp\left(-T_P/T_D\right)\right]$$

Minimum signal

$$P_{\min} = \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\frac{1}{\sqrt{2}\left[1-\exp\left(-2T_P/T_D\right)\right]} =$$

$$= \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\frac{1}{\sqrt{2}\left[1-1/e\right]}$$

That is, with $\sqrt{S_{v,b}} = 10\, nV/\sqrt{Hz}$ and $T_D = 10\mu s$

$$P_{\min} = \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\frac{1}{\sqrt{2}\left[1-1/e\right]} = \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\cdot 1{,}12 =$$

$$= 4{,}47\mu V \cdot 1{,}12 = 5{,}01\mu V$$

With respect to the optimum filtering, the result is worse by a factor

$$\frac{1{,}12}{1{,}07} = 1{,}05$$

The result obtained with the GI is definitely better than that obtained with the RC filter and the main reason is that the filtering of the signal by the GI is remarkably more near to that of the optimum filter.

B) Exponential rising pulse

Signal pulse waveform $\quad x(t) = P\exp\left(t-T_P\right)/T_D = P\exp-\left(T_P-t\right)/T_D \quad$ for $0<t<T_P$

Signal area $\quad A = PT_D\left[1-\exp\left(-T_P/T_D\right)\right]$

Normalized signal $\quad b(t) = \dfrac{x(t)}{A} = \dfrac{\exp\left(t-T_P\right)/T_D}{T_D\left[1-\exp\left(-T_P/T_D\right)\right]}$

B1) Optimum filtering

With respect to the case for the exponential decaying pulse reported in Sec. A1, the signal waveform has the same shape with time axis reversed and the noise is the same. Therefore, the computations of signal and noise are equal to those reported Sec. A1 for the exponential decaying pulse and the result is identical.

B2) Filtering with simple RC lowpass filter

In comparison with the case of decaying pulse treated in Sec. A2 we have the same weighting function and the same input noise, therefore the same output noise

Weighting function

$$w(t) = \frac{1}{T_F}\exp\left[\frac{t-T_P}{T_F}\right]$$

Weighting autocorrelation $k_{ww}(0) = \frac{1}{2T_F}$ which assuming $T_F = T_D$ reduces to $k_{ww}(0) = \frac{1}{2T_D}$

Output Noise Power $\overline{n_U^2} = S_{V,b} \cdot k_{ww}(0) = \frac{S_{V,b}}{2T_F} = \frac{S_{V,b}}{2T_D}$

The filtering of the signal, however, is remarkably better because in this case the signal and the weighting function have more similar waveform

$$x(t) = P\exp\left[(t-T_P)/T_D\right]$$

$$w(t) = \frac{1}{T_F}\exp\left[\frac{t-T_P}{T_F}\right]$$

Output signal

$$u_s = \int_0^{T_P} w(t)\, x(t)\, dt = \frac{P}{2}\left[1-\exp\left(-2T_P/T_D\right)\right]$$

S/N ratio

$$\left(\frac{S}{N}\right)_u = \frac{u_s}{\sqrt{\overline{n_u^2}}} = \frac{P}{\sqrt{S_{v,b}}}\sqrt{\frac{T_D}{2}}\left[1-\exp\left(-2T_P/T_D\right)\right] =$$

$$= \frac{P}{\sqrt{S_{v,b}}}\sqrt{\frac{T_D}{2}}\sqrt{1-\exp\left(-2T_P/T_D\right)} \cdot \sqrt{1-\exp\left(-2T_P/T_D\right)}$$

Minimum signal

$$P_{\min} = \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\frac{1}{\left[1-\exp\left(-2T_P/T_D\right)\right]} =$$

$$= \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\frac{1}{\left[1-e^{-2}\right]}$$

That is, with $\sqrt{S_{v,b}} = 10\, nV/\sqrt{Hz}$ and $T_D = 10\mu s$

$$P_{\min} = \sqrt{S_{v,b}}\sqrt{\frac{2}{T_D}}\frac{1}{\left[1-e^{-2}\right]} = 4{,}47\mu V \cdot 1{,}156 = 5{,}2\mu V$$

If we consider the factor of reduction of the performance of the RC filter with respect to the optimum filter in the two cases of exponential decaying pulse and exponential rising pulse, we note that the result obtained with the exponential rising pulse is remarkably better, as intuitively expected by observing the degree of similarity of the weighting function to the optimum in the two cases. In fact, for the exponential rising pulse filtered by the RC filter the factor of reduction of the performance is

$$\frac{1,156}{1,07} = 1,08$$

whereas it was 1,27 for the exponential decaying pulse.

B3) Filtering with a Gated Integrator (GI)

It can be readily verified that in this case the computations of filtered signal and noise are just the same as in the case of the exponential decaying signal filtered by a GI. Therefore, result obtained in Sec.A3 is valid also in the present case.