# PROBLEM 1

## Data summary

### Applied force F

rectangular pulse with duration $T_A$ = 80ms

### Piezoelectric force sensor

$A_q$=5pC/N force-to-charge conversion

$C_S$= 500pF total capacitance of sensor and connected circuit

### Preamplifier

Input resistance $R_{iA}$ very high > 500 MΩ , to be treated as $\infty$

$f_{pa}$=20MHz band limit

$\sqrt{S_v} = 4nV/\sqrt{Hz}$ wide-band (unilateral)

$\sqrt{S_i} = 0{,}01 pA/\sqrt{Hz}$ wide-band (unilateral);

where stated, consider in $S_i$ and in $S_v$ also a *1/f* component with corner frequency $f_c$=500 Hz

## A) Optimum filtering

Applied force F

piezoelectric charge $Q = A_Q F$

Preamp input signal $V_s = \frac{Q}{C_s}$

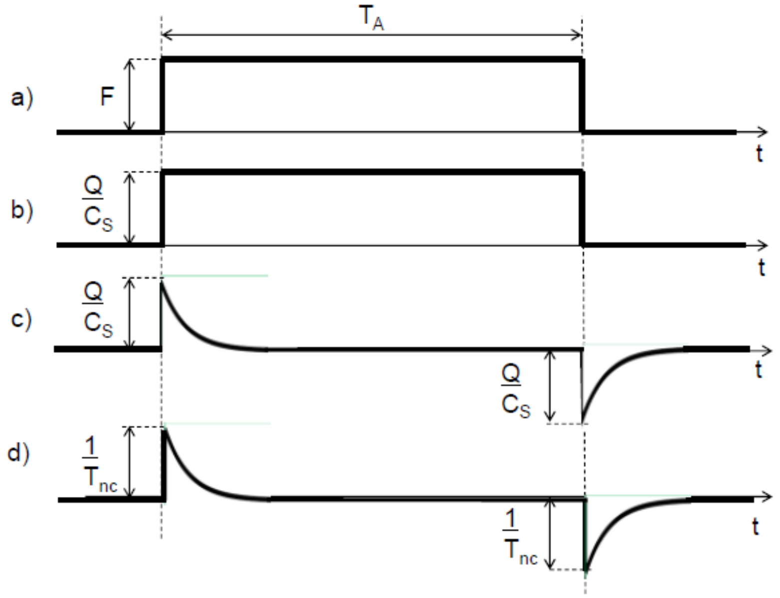


Fig. 1 a) Force applied to the piezoelectric sensor

b) Sensor voltage signal

c) output voltage signal of the whitening filter

d) weighting function of the matched filter

The noise at the preamp output is

$$S_S = S_v + \frac{S_i}{\omega^2 C_s^2} = S_v \frac{1+\omega^2 \frac{C_s^2 S_v}{S_i}}{\omega^2 \frac{C_s^2 S_v}{S_i}} = S_v \frac{1+\omega^2 T_{nc}^2}{\omega^2 T_{nc}^2}$$

the characteristic time constant is $T_{nc} = \frac{C_s S_v^{1/2}}{S_i^{1/2}} = 200\ \mu s$

The noise is NOT WHITE, therefore:

Optimum Filter = Whitening Filter plus Matched Filter.

<u>Whitening Filter $H_B(f)$</u>

It is a constant-parameter linear filter applied at the preamp output: a CR differentiator with time constant $T_B = T_{nc}$

$$\left|H_B\right|^2 = \frac{\omega^2 T_B^2}{1+\omega^2 T_B^2} = \frac{\omega^2 T_{nc}^2}{1+\omega^2 T_{nc}^2}$$

$H_B$ output: white noise

$$S_B = S_S \left|H_B\right|^2 = S_v$$

and signal composed by two exponential pulses (Fig.1), a positive one at t = 0 force start) and a negative one at $t = T_A$ (force end)

$$v_B(t) = \frac{Q}{C_S}\left[1(t)\cdot e^{-\frac{t}{T_{nc}}} - 1(t-T_A)\cdot e^{-\frac{t-T_A}{T_{nc}}}\right]$$

<u>Matched Filter</u>:

its weighting function $w_M$ has shape identical to the signal $v_B$ (Fig.1)

$$w_M(\alpha) = \frac{1}{T_{nc}}\left[1(t)\cdot e^{-\frac{\alpha}{T_{nc}}} - 1(\alpha-T_A)\cdot e^{-\frac{\alpha-T_A}{T_{nc}}}\right]$$

Since $T_{nc} << T_B$ the two exponential pulses are well separated (the overlap is negligible) and calculations are simplified. At the matched filter output

signal $$V_M = \int_0^\infty v_B(\alpha) w_M(\alpha)\, d\alpha$$

noise $$\overline{v_{n,M}^2} = \int_{-\infty}^{\infty} S_{V,b}\delta(\tau)\, k_{ww,M}(\tau)\, d\tau = S_{V,b}\; k_{ww,M}(0) = \frac{S_V}{2}\, k_{ww,M}(0)$$

In the case of a simple step force (force applied and maintained indefinitely) the output signal $v_B(\alpha)$ of the whitening filter and the corresponding weighting function $w_M(\alpha)$ would have just a positive exponential pulse and we would have

signal $$V_{M,u} = \frac{Q}{2C_s}$$

noise $$\sqrt{\overline{v_{n,Mu}^2}} = \frac{S_v^{1/2}}{\sqrt{4T_{nc}}}$$

therefore $$\left(\frac{S}{N}\right)_{Mu} = \frac{Q}{C_s}\cdot\frac{1}{\dfrac{S_v^{1/2}}{\sqrt{T_{nc}}}}$$

In the actual case, the applied force has finite duration $T_A$ (rectangular pulse) so that $v_B(\alpha)$ and $w_M(\alpha)$ include two well separate exponential pulses. Therefore, the signal amplitude and the noise mean square value are greater by a factor 2 than the case of simple step force so that S/N is improved by a factor $\sqrt{2}$

signal $$V_{M,r} = 2V_{M,u} = \frac{Q}{C_s}$$

noise $$\sqrt{\overline{v_{n,Mr}^2}} = \sqrt{2\overline{v_{n,Mu}^2}} = \frac{S_v^{1/2}}{\sqrt{2T_{nc}}}$$

therefore $$\left(\frac{S}{N}\right)_{Mr} = \frac{Q}{C_s}\cdot\frac{1}{\dfrac{S_v^{1/2}}{\sqrt{T_{nc}}}}\cdot\sqrt{2}$$

The minimum measurable voltage signal thus is

$$V_{M,r\min} = \sqrt{\overline{v_{n,Mr}^2}} = \frac{S_v^{1/2}}{\sqrt{2T_{nc}}} = 200nV$$

It corresponds to a minimum measurable piezoelectric charge

$$Q_{M,r\min} = C_S V_{M,r\min} = 10^{-16}\, C$$

and to a minimum measurable force

$$F_{M,r\min} = \frac{Q_{\min}}{A_Q} = 20\mu N$$

**B) Approximation of the optimum filtering with a Gated Integrator**

A weighting function that approximates the optimum filtering can be obtained with a gated integrator (GI) that weights in sequence the two exponential pulses of the signal $v_B$ and then combines the first and the second (negative) measurement.

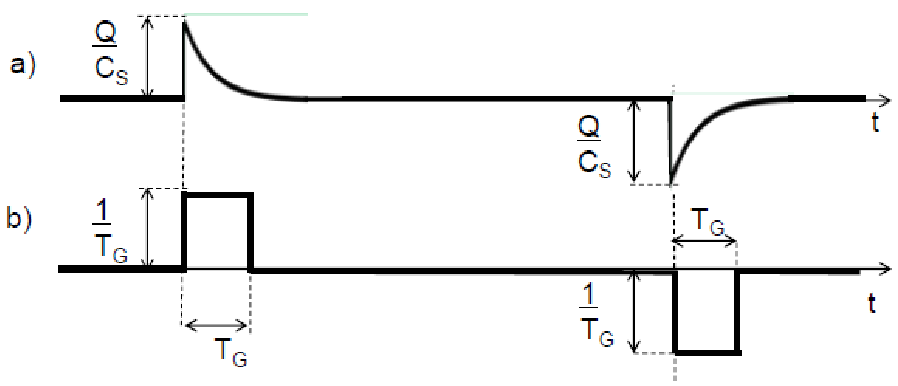


Fig. 2 a) Voltage signal at the whitening filter output
b) weighting function of the Gated Integrator

The result depends on the selected gating time $T_G$ . It is quite evident that for approximating the optimum weighting the duration $T_G$ must be comparable with the noise characteristic time constant $T_{nc}$ and it is possible to find out the $T_G$ value that gives the best result.

Also employing a GI we can gain a better insight by first considering the case of a simple step force and then that of rectangular force with finite duration $T_A$.

Denoting by $w_G$ the weighting function for the filtering by a GI, we have at the GI output

signal $$V_G = \int_0^\infty v_B(\alpha) w_G(\alpha)\, d\alpha$$

noise $$\sqrt{\overline{v_{n,G}^2}} = \int_{-\infty}^{\infty} S_{V,bil}\delta(\tau)\, k_{ww,G}(\tau)\, d\tau = S_{V,bil}\ k_{ww,G}(0) = \frac{S_V}{2}\, k_{ww,G}(0)$$

With a simple step force, considering a GI with normalized weight

$$w_G(\alpha) = \frac{1}{T_G} \quad \text{for} \quad 0 < \alpha < T_G \qquad \text{and elsewhere} \quad w_G(\alpha) = 0$$

We get

signal $$V_{Gu}=\frac{Q}{C_s}\cdot\frac{1-e^{-T_G/T_{nc}}}{\frac{T_G}{T_{nc}}}$$

noise $$\sqrt{\overline{v_{n,Gu}^2}}=\frac{S_v^{1/2}}{\sqrt{2T_G}}=\frac{S_v^{1/2}}{\sqrt{2T_{nc}}}\sqrt{\frac{T_{nc}}{T_G}}$$

therefore $$\left(\frac{S}{N}\right)_{Gu}=\frac{Q}{C_s}\cdot\frac{1}{\frac{S_v^{1/2}}{\sqrt{T_{nc}}}}\cdot\sqrt{2}\cdot\frac{1-e^{-T_G/T_{nc}}}{\sqrt{\frac{T_G}{T_{nc}}}}$$

that is, by denoting $x=\frac{T_G}{T_{nc}}$

$$\left(\frac{S}{N}\right)_{Gu}=\frac{Q}{C_s}\cdot\frac{1}{\frac{S_v^{1/2}}{\sqrt{T_{nc}}}}\cdot\sqrt{2}\cdot\frac{1-e^{-x}}{\sqrt{x}}=\left(\frac{S}{N}\right)_{Mu}\cdot\sqrt{2}\cdot\frac{1-e^{-x}}{\sqrt{x}}$$

the function $g(x)=\frac{1-e^{-x}}{\sqrt{x}}$ attains a maximum $g_{max}\approx 0{,}638$ for $x\approx 1{,}25$ ; therefore, at the maximum we get $\sqrt{2}g_{max}=0{,}9$ . In conclusion: with $T_G\approx 1{,}25\ T_{nc}\approx 250\ \mu s$ we get

$$\left(\frac{S}{N}\right)_{Gu}=\frac{Q}{C_s}\cdot\frac{1}{\frac{S_v^{1/2}}{\sqrt{T_{nc}}}}\cdot 0{,}9\cong\left(\frac{S}{N}\right)_{Mu}\cdot 0{,}9$$

That is, a result only 10% lower than that given by the optimum filter. We may further note that the setting of $T_G$ is not critical because the maximum of g(x) is not narrow: between x=1 and x=1,6 the function g(x) varies just by a few %.

In the case of force applied for a finite duration $T_A$ , $v_B(\alpha)$ has two well separated exponential pulses and $w_G(\alpha)$ has correspondingly two rectangular pulses of duration $T_G$. With respect to the case of simple step force, the signal value and the mean square noise value are increased by a factor 2 and hence the S/N by a factor $\sqrt{2}$ .

signal $$V_{Gr}=\frac{Q}{C_s}\cdot 2\frac{1-e^{-T_G/T_{nc}}}{\frac{T_G}{T_{nc}}}$$

noise $$\sqrt{\overline{v^2_{n,Gr}}} = \frac{S_v^{1/2}}{\sqrt{T_G}} = \frac{S_v^{1/2}}{\sqrt{T_{nc}}}\sqrt{\frac{T_{nc}}{T_G}} = 253 \text{ nV}$$

hence $$\left(\frac{S}{N}\right)_{Gr} = \frac{Q}{C_s}\cdot\frac{1}{\frac{S_v^{1/2}}{\sqrt{T_{nc}}}}\cdot 2\cdot\frac{1-e^{-T_G/T_{nc}}}{\sqrt{\frac{T_G}{T_{nc}}}} = \left(\frac{S}{N}\right)_{Mr}\sqrt{2}\cdot\frac{1-e^{-T_G/T_{nc}}}{\sqrt{\frac{T_G}{T_{nc}}}}$$

The maximum is attained with $T_G/T_{nc} \approx 1{,}25$, which gives

$$\left(\frac{S}{N}\right)_{Gr} \cong \left(\frac{S}{N}\right)_{Mr}\cdot 0{,}9$$

thus a minimum measurable piezoelectric charge

$$Q_{G,r\min} = \frac{Q_{M,r\min}}{0{,}9} = 1{,}11\cdot 10^{-16} C$$

and a corrisponding minimum measurable frce

$$F_{g,r\min} = \frac{F_{M,r\min}}{0{,}9} = 22\mu N$$

**C) <u>Effect of the 1/f noise</u>**

C1) <u>Current noise</u>

$$S_{if} = \frac{S_i f_c}{f}$$

It is filtered (integrated) by the capacitance $C_S$ and transformed in voltage noise spectrum

$$S_{if}\frac{1}{\omega^2 C_S^2}$$

The noise whitening filter then cancels the pole at $\omega=0$ and replaces it with a pole at $\omega_{nc}=1/T_{nc}$

$$S_{if}\frac{1}{\omega^2 C_S^2}\frac{\omega^2 T_{nc}^2}{1+\omega^2 T_{nc}^2} = S_{if}\frac{T_{nc}^2}{C_S^2}\frac{1}{1+\omega^2 T_{nc}^2} = S_i\frac{f_c}{f}\frac{T_{nc}^2}{C_S^2}\frac{1}{1+\omega^2 T_{nc}^2}$$

Taking into account the equation $T_{nc} = \frac{C_s S_v^{1/2}}{S_i^{1/2}}$ the output of the noise whitening filter is

$$S_i\frac{f_c}{f}\frac{T_{nc}^2}{C_S^2}\frac{1}{1+\omega^2 T_{nc}^2} = S_i\frac{f_c}{f}\frac{S_v}{S_i}\frac{1}{1+\omega^2 T_{nc}^2} = S_v\frac{f_c}{f}\frac{1}{1+\omega^2 T_{nc}^2} = S_V\frac{f_c}{f}\frac{1}{1+\left(2\pi f\right)^2 T_{nc}^2}$$

We recognize that the noise brought to the output of the whitening filter by the current 1/f noise is equal to the voltage 1/f spectrum filtered by a low-pass filter with a simple pole at $\omega_{nc}=1/T_{nc}$

$$S_V \frac{f_c}{f} \frac{1}{1+\left(2\pi f\right)^2 T_{nc}^2}$$

The processing carried out with the Gated Integrator and illustrated in Sec.B performs

- A low-pass filtering with band-limit frequency $1/2T_G$ (integration over the gate time $T_G$ )
- A high-pass filtering with band-limit frequency $1/T_A$ (subtraction of two GI operations separated by a time interval $T_A$ )

The low-pass filtering brings the upper frequency limit down to $f_s \approx 1/2T_G$ , which by selecting $T_G \approx 1{,}25 T_{nc}$ is set to $f_s \approx 1/2{,}5\ T_{nc}$ = 2kHz.

The high-pass filtering sets a lower band-limit at $f_{ii} \approx 1/T_A \approx$ 12,5Hz.
This 1/f noise contribution can thus be evaluated with the sharp-cut in frequency

$$\sqrt{\overline{v_{n,Gf}^2}} = \sqrt{S_v f_c} \cdot \sqrt{\ln\left(\frac{f_s}{f_{ii}}\right)} \approx \sqrt{S_v f_c} \cdot \sqrt{\ln\left(\frac{T_A}{2{,}5 T_{nc}}\right)} = 201 \text{ nV}$$

and turns out to be tolerable, since it is slightly lower than the contribution of the white noise components

$$\sqrt{\overline{v_{n,Gr}^2}} = \frac{S_v^{1/2}}{\sqrt{T_G}} = \frac{S_v^{1/2}}{\sqrt{1{,}25 T_{nc}}} = 253 \text{ nV}$$

C1) <u>Voltage noise</u>

$$S_{Vf} = \frac{S_V f_c}{f}$$

For the voltage noise 1/f component

- The sensor capacitance $C_S$ does not introduce any filtering
- The noise whitening filter gives a high-pass filtering which attenuates the spectrum in the low frequency region. It enforces to the voltage 1/f noise component a lower band limit $f_{iV} \approx 1/2\pi T_{nc} \approx$ 800Hz much stronger than that enforced to the current 1/f noise component $f_{ii} \approx 1/T_A \approx$ 12,5Hz

$$S_{Vf} \frac{\omega^2 T_{nc}^2}{1+\omega^2 T_{nc}^2} = S_V \frac{f_c}{f} \frac{\left(2\pi f\right)^2 T_{nc}^2}{1+\left(2\pi f\right)^2 T_{nc}^2}.$$

- The integration carried out with the Gated Integrator (illustrated in Sec.B) attenuates the spectrum in the high frequency region and sets an upper bandlimit $f_s \approx 1/2T_G$.
- The double measurement by the GI sets also the high-pass band-limit $f_{ii} \approx 1/T_A \approx 12{,}5$Hz as for the current 1/f noise component, which however in this case has negligible effect since it is at much lower frequency than the high-pass band-limit produced by the whitening filter.

In summary, the contributions of the voltage 1/f component and of the current 1/f noise component are computed with the same spectral density $S_{Vf} = \frac{S_V f_c}{f}$ filtered with the same upper bandlimit $f_s \approx 1/2T_G$ , but with a much higher band-limit at low-frequency for the voltage noise. We can conclude that the contribution of the voltage 1/f noise is much lower than that of the current 1/f noise and is therefore negligible