PROBLEM 2

Data summary

$v_s=V_s \cos\omega_s t$ signal

V_s amplitude that slowly varies over intervals ≥ 1s

$f_s = 2kHz$ signal frequency (angular frequency $\omega_s=2\pi f_s$)

$v_R=B\cos\omega_s t$ reference signal, synchronous with the signal to be measured

$f_o= f_s =1kHz$ resonance frequency of the tuned filter

$Q=5$ quality factor of the tuned filter

$\sqrt{S_v} = 100 nV/\sqrt{Hz}$ effective power density (unilateral) of wide-band noise

$f_c = 500Hz$ corner frequency of the $1/f$ noise

(A) Filtering with tuned filter RLC

A1 –Characteristic parameters of the filter

- **Resonance frequency** (characteristic oscillation frequency of the filter) $\omega_o = \frac{1}{\sqrt{LC}}$
- **Damping factor** of the response in time $\alpha_o = \frac{1}{2RC}$
- **Quality** Factor $Q = \frac{\omega_o}{2\alpha_o}$
- Unity gain at the resonance frequency f_o of the filter stage $H(f_o)=1$
(the RLC filter is inserted in an amplifier circuit)

NB: since the values of ω_o and Q are known, the value of $\alpha_o = \omega_o/2Q$ is also known

A2 – Weighting function in time

***h(t)* pulse-response**: oscillation with exponentially damped amplitude

$$h(t) = 1(t)\, A e^{-\alpha_o t} \cos\omega_o t$$

(the amplitude A is a known function of f_o and Q. It is not reported here because it is not necessary for the computations in this problem)

Weighting function $w_o(t)$ for a measurement of the filter output amplitude obtained by sampling at the instant t_m

$$w_o(t) = h(t_m - t) = 1(t_m - t)\, A e^{-\alpha_o(t_m - t)} \cos\omega_o (t_m - t)$$

(for constant-parameter filter $w_o(t)$ is the δ-response translated at time t_m and inverted in time).

If the sampling instant t_m is shifted, also the weighting function is shifted, hence its peaks are shifted with respect to the sinusoidal signal.

A3 – Transfer function in frequency *H(f)* or **Weighting function in frequency $W_o(f)$)**

- The main part is centered at the resonance frequency f_o
  and $|H(f)|^2$ is well approximated with a parabolic function.
- At resonance frequency the phase is zero (the output signal is aligned with the input signal).
- At resonance frequency the gain is unity, the output signal has amplitude equal to the input signal
- The equivalent bandwidth for computing the rms noise at the filter output is

$$\Delta f_{no} = \frac{\pi}{2}\frac{f_o}{Q} = 314 Hz$$

A4 – Evaluation of noise and minimum measurable signal

Denoting by S_T the noise total spectral density (white noise plus *1/f* noise)

$$S_T\left(f_o\right) = S_v + S_v \frac{f_c}{f_o} = 1{,}5 \cdot S_v \qquad \text{that is} \qquad \sqrt{S_T\left(f_o\right)} = \sqrt{1{,}5}\sqrt{S_v} = 122\, nV\big/\sqrt{Hz}$$

The noise rms value is

$$\sqrt{\overline{v_{n,o}^2}} \approx \sqrt{S_T\left(f_o\right)} \cdot \sqrt{\Delta f_{n,o}} = \sqrt{1{,}5} \cdot \sqrt{S_v} \cdot \sqrt{\frac{\pi}{2}\frac{f_o}{Q}} \approx 2{,}17 \mu V$$

At centerband the signal gain is unity and the S/N is

$$\left(\frac{S}{N}\right)_o = \frac{V_s}{\sqrt{\overline{v_{n,o}^2}}}$$

The minimum measurable signal (for which S/N=1) is

$$V_{s\min,o} = \sqrt{\overline{v_{n,o}^2}} = 2{,}17 \mu V$$

A5 – Correct execution of the measurement of the filter output

A correct measurement of V_s is obtained by sampling the filter output at a properly selected time t_m , that is, a time that corresponds to a maximum of the output signal, which is oscillatory as the input signal. This can be obtained also in case of a signal with very small amplitude by using the reference signal for synchronizing the output sampling.

This selection of the sampling time is confirmed also by analyzing the action of the weighting function. The measurement is obtained multiplying the input signal by the weight and integrating the product, therefore it is clear that the result is maximized (hence the S/N is maximized) by aligning the weighting function with the input signal.

The variations of the signal amplitude occur over time intervals of ≈1s or longer, therefore in the Fourier domain the variation of the signal amplitude is described by significant frequency components up to a maximum frequency of about 1Hz. In order to monitor correctly the signal amplitude variation it is necessary :

a) to employ a sampling frequency double of the maximum frequency, that is, at least 2Hz, with an interval between samples of ≈500ms or shorter

b) to avoid that the measurement of a sample include contributions of previous samples. Therefore, the weighting function must decrease to negligible level in a time shorter than the interval between the samples, that is, shorter than ≈500ms.

The weighting function decreases exponentially with the time interval from the sampling time t_m

$$1(t_m - t)e^{-\alpha_o(t_m - t)} = 1(t_m - t)e^{-(t_m - t)/T_d}$$

where

$$T_d = \frac{1}{\alpha_o} = \frac{2Q}{\omega_o} = \frac{Q}{\pi f_o} \approx 1,6ms$$

The weighting function is reduced to negligible level after a time interval $t_m - t = 5T_d \approx 8ms$ that is, a time much shorter than the interval of 500ms between samples

(B) <u>Filtering with a Lock-in Amplifier (LIA)</u>

B1 – Weighting function of the LIA

The LIA is a time-variant filter. Its basic stages are an analog multiplier (that multiplies input signal and reference signal) and a low-pass filter. The low-pass filter is a simple pole RC filter with time constant T_F and weighting function

$$w_F(t) = 1(t_m - t)e^{-(t_m - t)/T_F}$$

The LIA weighting function is the product of the reference signal and of the low-pass weighting function

$$w_L(t) = B\cos\omega_o t \cdot w_F(t) = 1(t_m - t)e^{-(t_m - t)/T_F} B\cos\omega_o t$$

The measurement is carried out by sampling the LIA output at the time t_m where the measurement has to be taken. We may note that:

a) if the sampling time t_m is shifted the exponential damping function is correspondingly shifted, but the maxima and minima are not shifted, they stay synchronized with the reference signal.

b) the damping is ruled by the time constant T_F .

B2 – LIA parameter selection

In order to have a correct measurement of the variation of the amplitude V_s with time, it is necessary to avoid that the measurement of a sample include contributions of previous samples. Therefore, the weighting function must decrease to negligible level in a time shorter than the interval between the samples, that is, shorter than ≈500ms. Therefore, the time constant T_F must ensure a sufficient damping: for instance, in order to reduce the weight to < 1/100 it is necessary to have $5T_F < 500ms$. For instance, with T_F =50ms we have a band-limit of the low-pass filter of the LIA

$$f_{Fn} = \frac{1}{4T_F} = 5Hz$$

B3 – Minimum measurable signal

Employing the sinusoidal reference in phase with the signal, the $(S/N)^2$ obtained is given by the ratio of the signal power ($V_s^2/2$) and one half of the noise power (in-phase noise component) in the band width $2f_{Fn}$ defined by the low-pass filter centered on the reference frequency $f_o = f_s$.

$$\left(\frac{S}{N}\right)_L = \frac{V_s}{\sqrt{2S_T f_{Fn}}}$$

The minimum measurable signal therefore is

$$V_{s\min,L} = \sqrt{2S_T f_{Fn}} = 387nV$$

B4 – Correct execution of the measurement of the LIA output

<u>It is not necessary to synchronize the sampling time t_m with the reference signal.</u> We already noted that by shifting t_m the maxima and minima of the weighting function remain synchronized with the reference, the output is a slowly varying signal that follows the slow variations of the amplitude V_S.

The time constant T_F of the low-pass filter of the LIA has already been selected in B2 to ensure that by sampling the output at intervals of 500ms the measurement of a sample does not receive contributions from other samples.

B5 – Comparison of LIA and resonant filter

The performance obtained with a LIA is clearly superior to that obtained with a resonant filter for various basic reasons.

- in a LIA it is not necessary to synchronize the sampling of the LIA output, because the action of the multiplier already ensures the synchronization of the filtering operation with the reference.
- in a LIA it is possible and practical to maintain the passband centered on the signal even if the frequency of the signal varies and even if the apparatus works in variable ambient conditions (variable temperature, etc.). With tuned filters such variations cause a misalignment of the filtering passband with respect to the signal frequency, whereas in LIA the passband remains centered on the reference frequency
- in a LIA it is possible and practical to obtain a much narrower filtering band-pass in comparison to resonant filters, because it is set by the band-limit of a simple low-pass filter. In resonant filters the resonant bandpass is limited by the maximum value of the quality factor Q actually available in practice, which is subject to strong limitations depending on the resonant frequency value, particularly severe at low frequency.