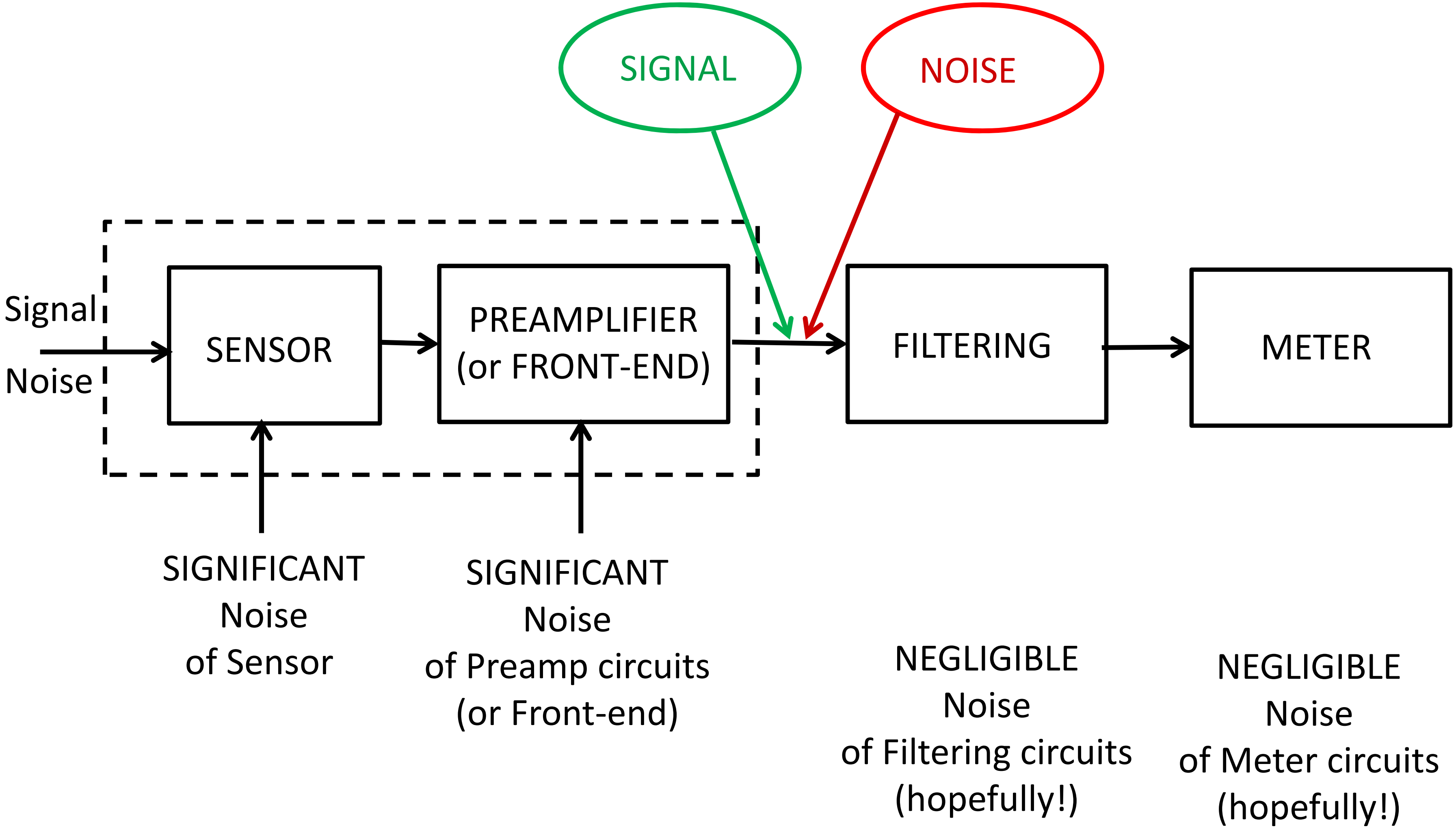
Sensors, Signals and Noise

COURSE OUTLINE

- Introduction
- Signals and Noise
- Filtering
- Sensors and associated electronics

Set-Up for Sensor Measurements



Mathematical Description of Signals

- Time domain and frequency domain analysis
- Energy signals and correlation functions
- Energy Spectrum
- Power signals, Correlation Functions and Power Spectrum
- Note on truncated signals

and for those who trust only analytical demonstrations

- Appendix: Crosscorrelation and Convolution

Time domain and frequency domain analysis of signals

Signals: mathematical description

- Signals = electric variables *x* (voltage, current ...) that carry information
- In the domain of time *t* : deterministic functions *x = x(t)*

Example: exponential pulse

$$x = 1(t)e^{-t/T}$$

In the domain of time t can be considered
linear superposition (sum) of elementary δ-pulses of amplitude (i.e. area) *x(t)dt*

In the domain of frequency *f* (Fourier transform domain) can be considered
linear superposition (sum) of elementary sinusoid components

Signals: mathematical description

linear superposition (sum) of elementary sinusoid components

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{i2\pi f t} \, df$$

$$X(f) = F[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-i2\pi f t} \, dt$$

- *X(f)* = Fourier transform of x(t)
- *X(f)* is complex : Module and Phase (or Real and Imaginary parts)

RECALL: since *x(t)* is real, the transform *X(f)* has simple and useful properties, for instance:

$X(-f) = X^*(f)$, that is $|X(-f)| = |X(f)|$ and $\arg[X(-f)] = -\arg[X(f)]$

...... and various other properties!

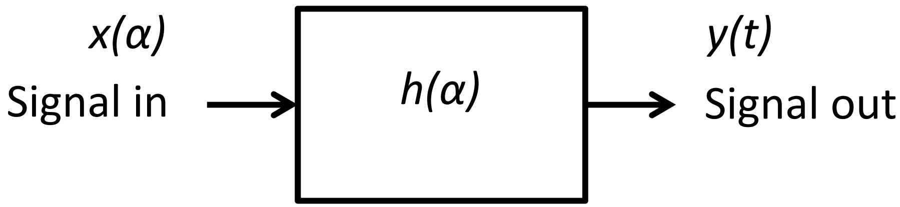
Convolution

Constant-parameter linear filters (NO switches, NO time-variant components!!)

are characterized by

$H(f)$ transfer function in frequency domain $\qquad H(f) = F[h(t)]$

$h(t)$ δ-response in time domain $\qquad h(t) = F^{-1}H(f)]$



The input x(α) can be described as a linear superposition (sum) of elementary δ-pulses of amplitude x(α)dα

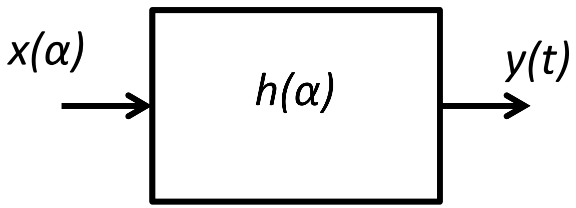
therefore

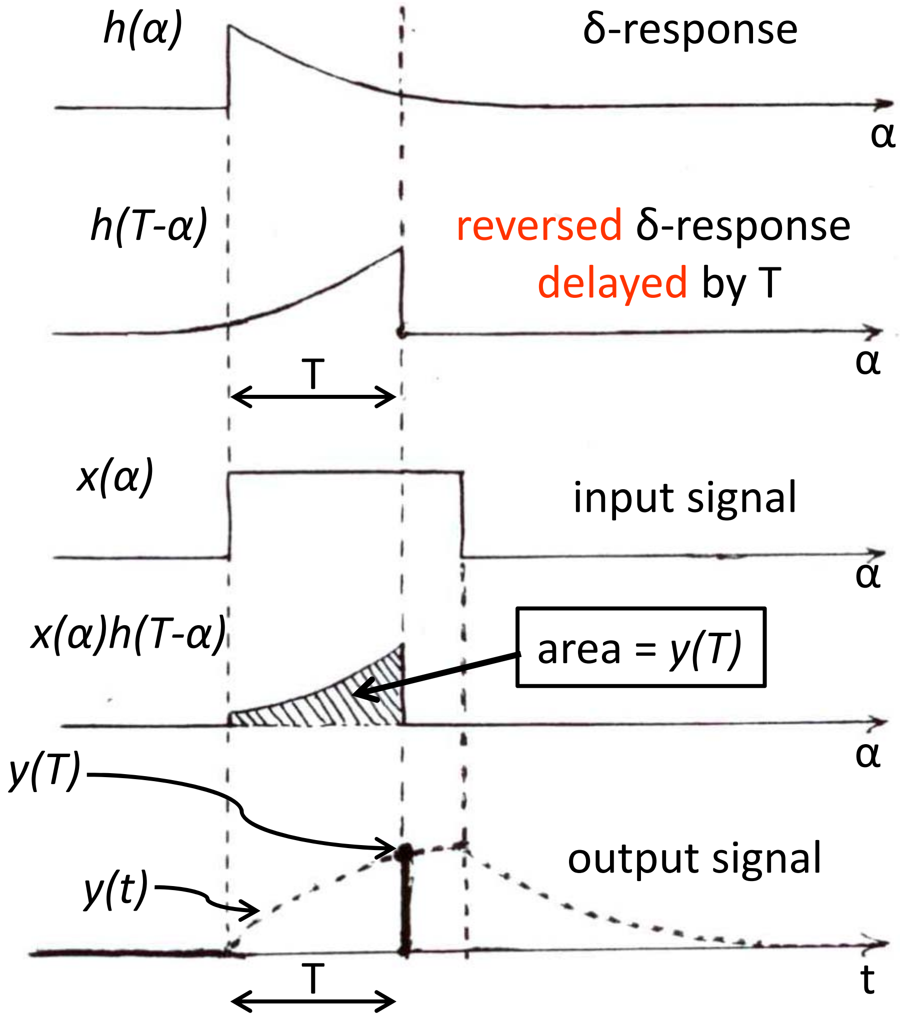
the output *y(t)* can be described as a linear superposition (sum) of elementary δ-pulse responses *x(α)dα h(t-α)*

$$y(t) = x(\alpha) * h(\alpha) = \int_{-\infty}^{+\infty} x(\alpha)h(t-\alpha)d\alpha$$

Computing the convolution

$$y(t) = \int_{-\infty}^{+\infty} x(\alpha)h(t-\alpha)d\alpha$$





Energy signals and correlation functions

Signal Energy

The Energy *E* of a signal *x(t)* is defined as

$$E = \lim_{T\to\infty} \int_{-T}^{T} x^2(\alpha)d\alpha = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha$$

Signals *x(t)* with finite *E* are called <u>energy-signals</u>. Typical example: <u>pulse signals</u>

INTUITIVE VIEW OF ENERGY:

Let *x(t)* be a voltage pulse on a unitary resistance *R=1 Ω*

then *E* is the energy dissipated in *R* by the pulse

Signal Auto-Correlation Function (Energy-type)

$$k_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} x(\alpha)x(\alpha+\tau)d\alpha = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

$k_{xx}(\tau)$ gives the degree of similarity of $x(t)$ with itself <u>shifted by τ</u>

Energy = Autocorrelation <u>at zero-shift</u>

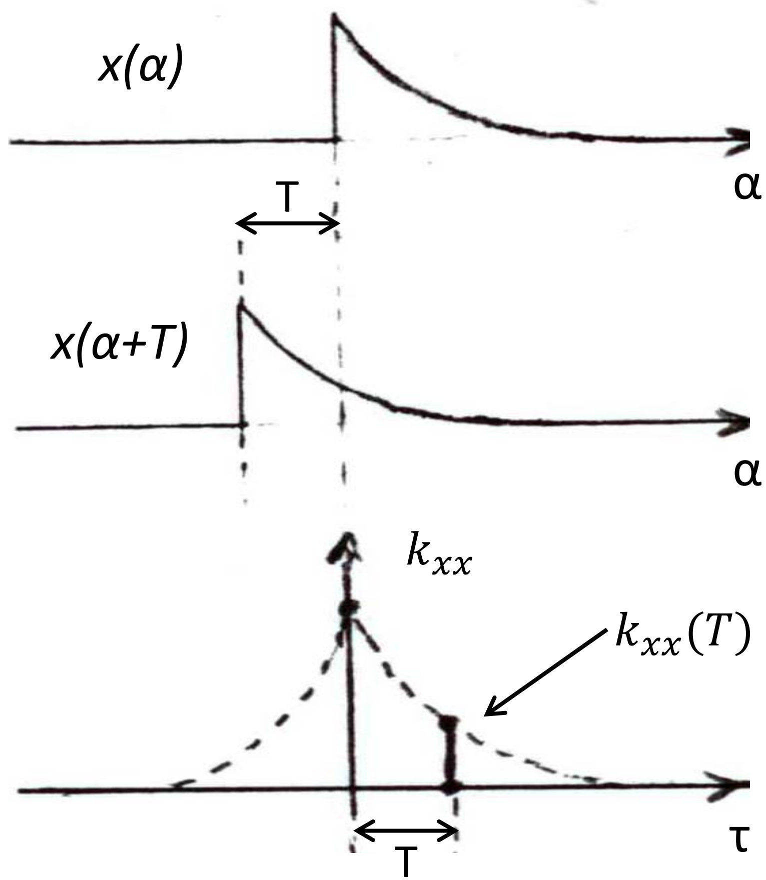
$$k_{xx}(0) = E$$

Signal Auto-Correlation Function (Energy-type)

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

Case: single pulse (exponential)

$$x = 1(t)\,Ae^{-t/T_P}$$



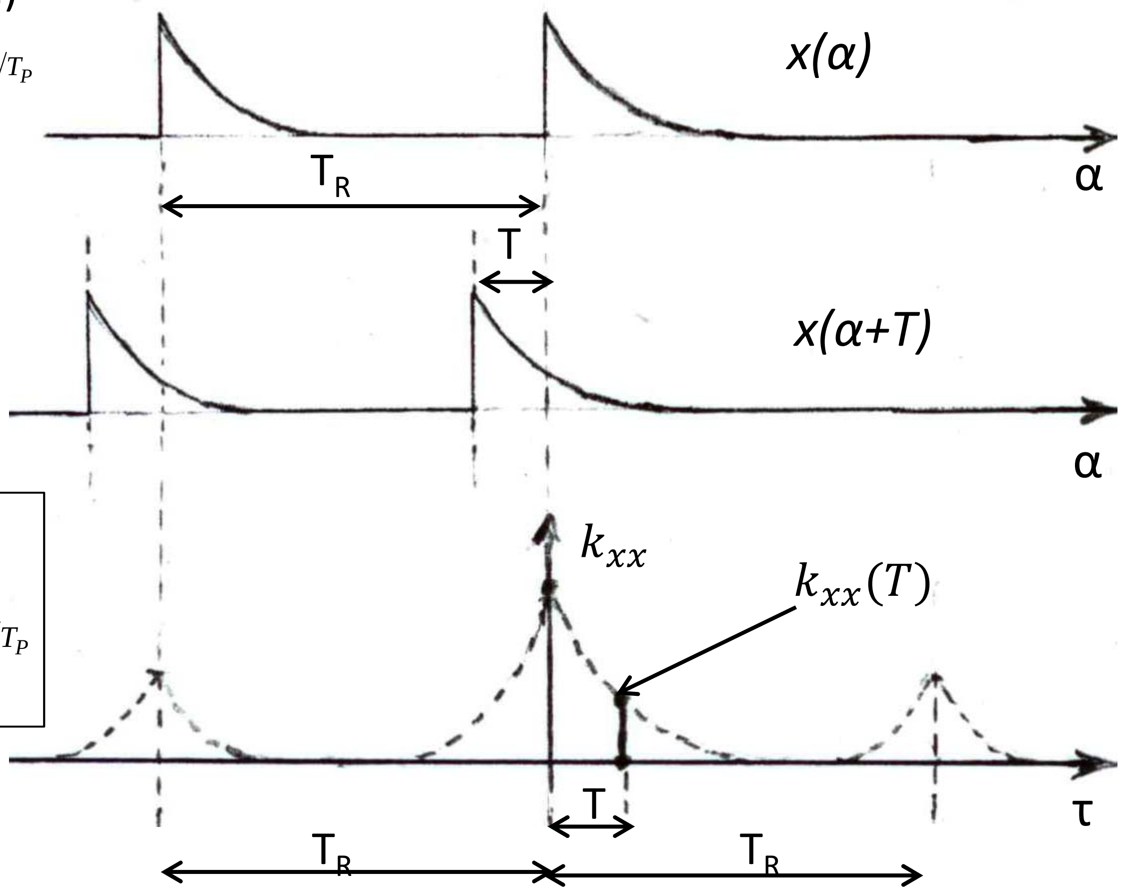
$$\boxed{k_{xx}(\tau) = A^2\,\frac{T_P}{2}\,e^{-|\tau|/T_P}}$$

Signal Auto-Correlation Function (Energy-type)

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

Case: double pulse (exponential)

$$x = 1(t)\,Ae^{-t/T_P} + 1(t-T_R)\,Ae^{-(t-T_R)/T_P}$$



$$k_{xx}(\tau) = A^2 T_P e^{-|\tau|/T_P} +$$
$$+A^2\frac{T_P}{2}e^{-|\tau-T_R|/T_P} + A^2\frac{T_P}{2}e^{-|\tau+T_R|/T_P}$$

Signal Cross-Correlation Function (Energy-type)

$$k_{xy}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} x(\alpha)y(\alpha+\tau)d\alpha = \int_{-\infty}^{\infty} x(\alpha)y(\alpha+\tau)d\alpha$$

- x(t) and y(t) are **two different** signals of energy-type

- $k_{xy}(\tau)$ gives the degree of similarity of *x(t)*

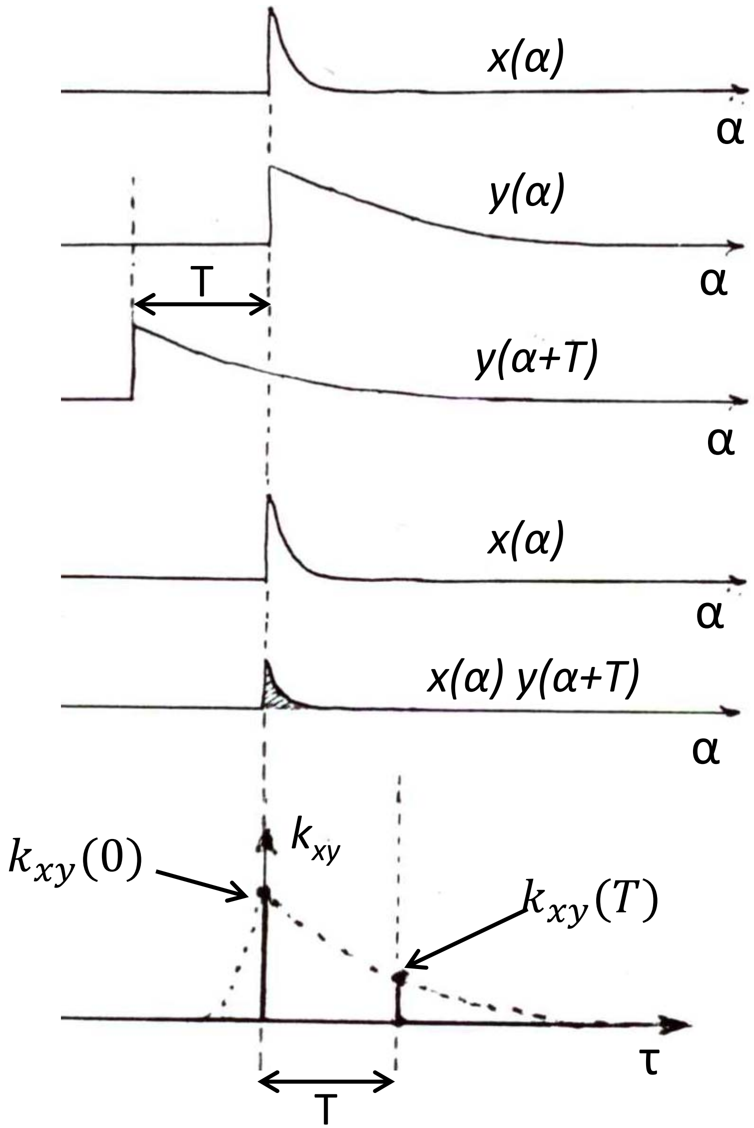
with *y(t)* shifted by τ to left (towards earlier time)

- Various denominations for $k_{xy}(\tau)$ *:*

Cross-Correlation function of x and y

Mutual-Correlation function of x and y

Signal Cross-Correlation Function (Energy-type)



Building step-by-step

the Cross-Correlation function

$$k_{xy}(\tau) = \int_{-\infty}^{\infty} x(\alpha) y(\alpha+\tau) d\alpha$$

Cross-Correlation obtained by Convolution

Convolution

$$x * y = z(T)$$

is different from Crosscorrelation $k_{xy}(T)$

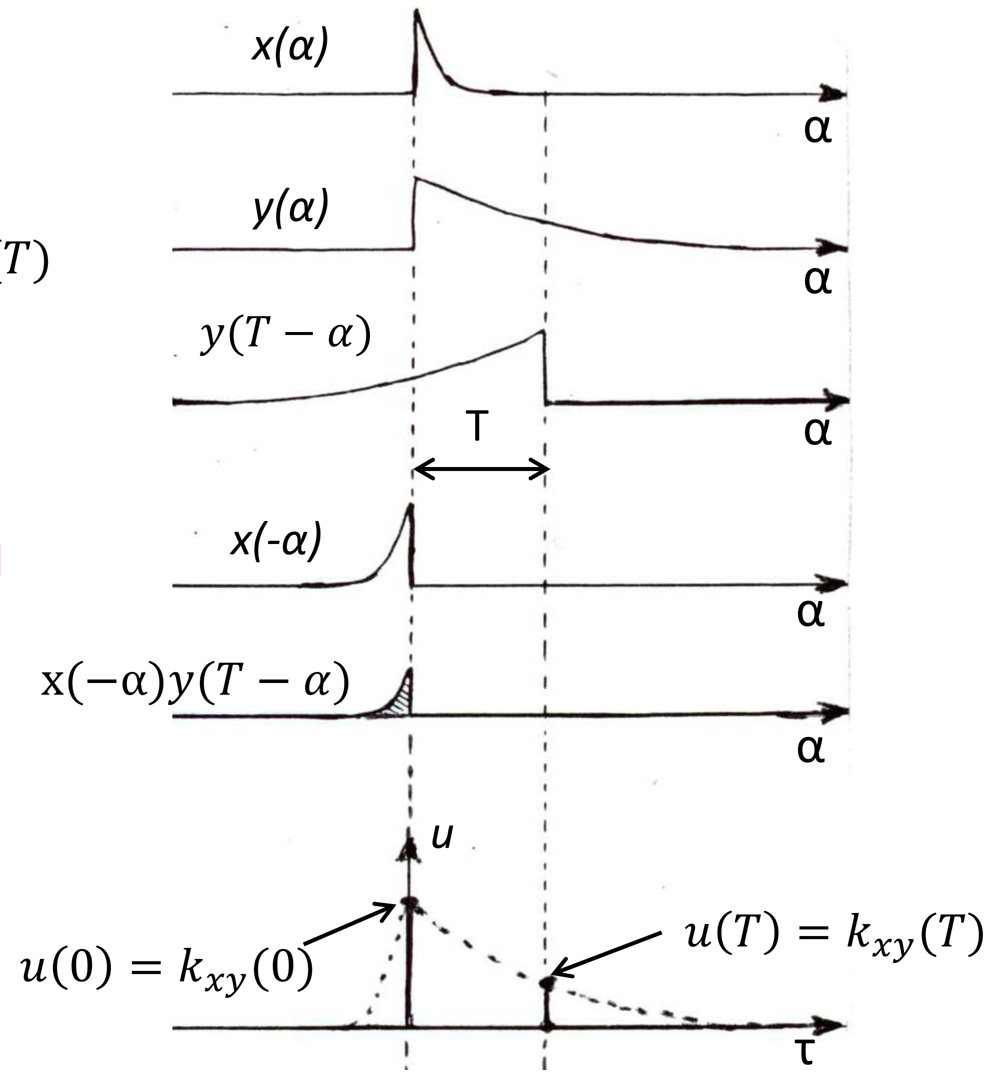
However

Convolution with **first term reversed**

$$x(-a) * y(a) = u(T)$$

is equal to Crosscorrelation

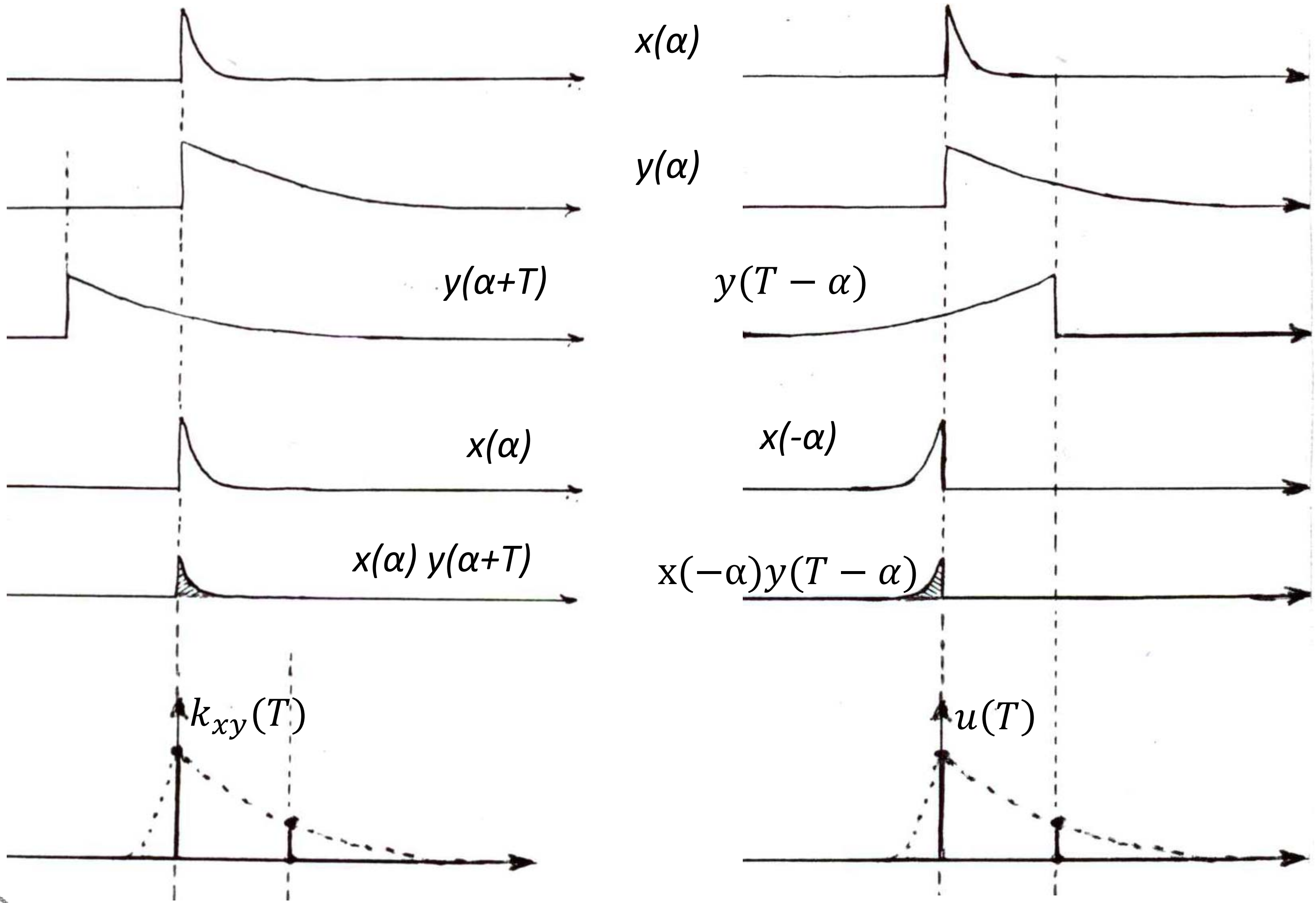
$$u(T) = k_{xy}(T)$$



Cross-Correlation obtained by Convolution

$k_{xy}(T)$

$u(T) = x(-a) * y(a)$



Energy Spectrum

Energy Spectrum

Energy signal *x(α)* with Fourier transform X(f): by Parseval's theorem

$$E = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha = \int_{-\infty}^{\infty}|X(f)|^2 df = 2\int_0^{\infty}|X(f)|^2 df$$

$S_x(f) = |X(f)|^2$ is called the Energy Spectrum of the signal *x(α)*

INTUITIVE VIEW OF ENERGY SPECTRUM:

Let *x(t)* be voltage on a unitary resistance *R=1 Ω*

power = voltage *x(t)* multiplied by current *x(t)*

x(t) = sum of sinusoid components with frequency f and amplitude $|X(f)|df$

sinusoids are orthogonal functions

No power from multiplication of voltage and current of **different components** (different f)

Every component (at frequency f) contributes an energy

$$dE = |X(f)|^2 df + |X(-f)|^2 = 2\,|X(f)|^2 df$$

Energy Spectrum

- Alternative definition of the Energy Spectrum is

$$S_x = F[k_{xx}]$$

- Knowing that $k_{xx} = x(-\alpha) * x(\alpha)$ we see that the two definitions are consistent

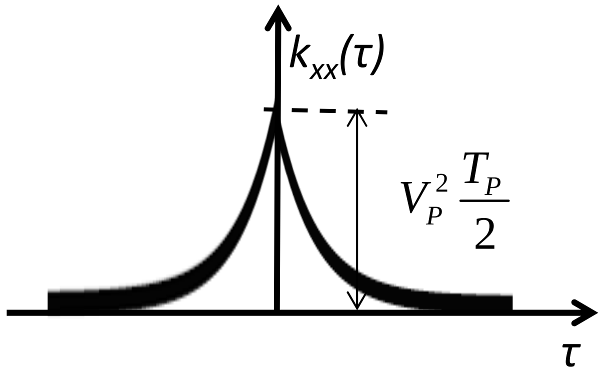
$$S_x = F[k_{xx}] = F[x(-\alpha) * x(\alpha)] = X(-f)X(f) = X^*(f)X(f) = |X(f)|^2$$

and by a basic property of Fourier transforms

$$\int_{-\infty}^{\infty} S_x(f)df = \int_{-\infty}^{\infty} |X(f)|^2 df = k_{xx}(0) = E$$

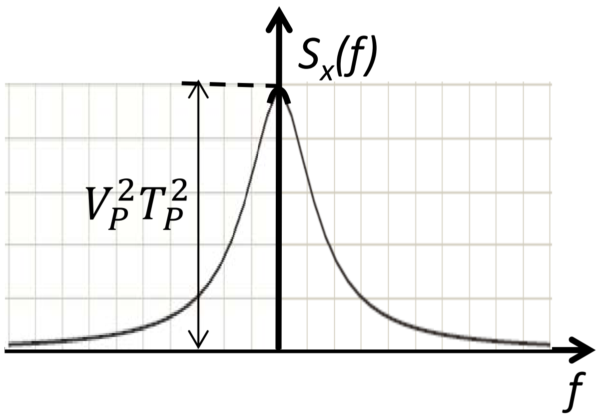
Example of
Energy, Autocorrelation and Energy Spectrum

Exponential pulse $$V_P \exp\left(-\frac{t}{T_P}\right) = x(t) \rightleftarrows X(f) = V_P T_P \frac{1}{1 + j2\pi f T_P}$$



Autocorrelation function $$k_{xx}(\tau) = \frac{V_P^2 T_P}{2} \exp\left(-\frac{\tau}{T_P}\right)$$

Energy $$E = k_{xx}(0) = V_P^2 \frac{T_P}{2}$$



Energy Spectrum $$S_x(f) = \left|X(f)\right|^2 = \frac{V_P^2 T_P^2}{1 + \left(2\pi f T_p\right)^2}$$

Energy $$E = \int_{-\infty}^{\infty} S_x(f)\, df = V_P^2 \frac{T_P}{2}$$

Signal **Auto**-Correlation Function (Energy-type)

- k_{xx} is **symmetrical**: $k_{xx}(\tau) = k_{xx}(-\tau)$
- k_{xx} has **positive maximum at zero** shift: $\boldsymbol{k_{xx}(0) > |k_{xx}(\tau)|}$ with $\boldsymbol{k_{xx}(0) > 0}$

Signal **Cross**-Correlation Function (Energy-type)

- x(t) and y(t) are two different energy-type signals
- k_{xy} is NOT symmetrical, however $\boldsymbol{k_{xy}(\tau) = k_{yx}(-\tau)}$
- the maximum of k_{xy} is neither necessarily positive nor at zero shift, however, the absolute maximum value is limited

$\left|k_{xy}(\tau)\right| \leq \frac{1}{2}\,[k_{xx}(0) + k_{yy}(0)]$ (**linear mean** of the maxima of k_{xx} and k_{yy})

$\left|k_{xy}(\tau)\right| \leq \sqrt{k_{xx}(0)\, k_{yy}(0)}$ (**geometric mean** of the maxima of k_{xx} and k_{yy})

Auto-Correlation of sum-signals

The autocorrelation of the sum of two signals x(t) and y(t)

$$k_{zz}(\tau) = \int_{-\infty}^{\infty} [x(\alpha) + y(\alpha)][x(\alpha+\tau) + y(\alpha+\tau)]d\alpha$$

is the sum of their auto- and cross-correlations

$$k_{zz}(\tau) = k_{xx}(\tau) + k_{xy}(\tau) + k_{yx}(\tau) + k_{yy}(\tau)$$

The energy spectrum $S_z(f) = F[k_{zz}(\tau)] = |Z(f)|^2$

is the sum of the two SPECTRA (real) and of the two CROSS-SPECTRA (complex conjugate)

$$S_z(f) = |X(f)|^2 + X^*(f)Y(f) + X(f)Y^*(f) + |Y(f)|^2$$

$$S_z(f) = S_x(f) + S_{xy}(f) + S_{yx}(f) + S_y(f)$$

Complex conjugate

Power signals, Correlation Functions and Power Spectrum

Signal Power

For signals *x(t)* that have NOT finite energy $E \rightarrow \infty$ (DC, sinusoids, periodic signals, etc.) the Power *P* is defined as the time-average

$$P = \lim_{T\to\infty} \int_{-T}^{T} \frac{x^2(\alpha)}{2T} d\alpha$$

Parseval theorem is valid for the entire integral $\int_{-\infty}^{+\infty}$

but NOT for the truncated integral $\int_{-T}^{+T}$

For computing *P* in *f* domain instead of truncated integral we use truncated signal $x_T(t)$

$$x_T(\alpha) = x(\alpha) \qquad \text{for } |\alpha| \leq T$$
$$x_T(\alpha) = 0 \qquad \text{for } |\alpha| > T$$

We can thus exploit Parseval theorem: with $X_T(f) = F[x_T(\alpha)]$ we get

$$P = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{{x_T}^2(\alpha)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{2T} df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T} df$$

The Power Spectrum of the signal *x(α)* is defined as the integrand

$$S_x(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T} \qquad \text{and} \qquad P = \int_{-\infty}^{\infty} S_x(f) df$$

Signal Auto-Correlation Function (Power-type)

Just like power P, the autocorrelation of power signals is defined as **time-average**

$$K_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha \qquad \text{note that} \qquad P = K_{xx}(0)$$

Also here we use truncated signal *$x_T(\alpha)$* instead of truncated integral $\int_{-T}^{T}$

$$K_{xx}(\tau)) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{x_T(\alpha)x_T(\alpha+\tau)}{2T} d\alpha$$

<u>NB1</u>: for finite T it is $\int_{-T}^{T} x(\alpha)x(\alpha+\tau)d\alpha \neq \int_{-\infty}^{\infty} x_T(\alpha)x_T(\alpha+\tau)d\alpha$

but for $\lim_{T\to\infty}$ the $=$ is valid

<u>NB2</u>: *$x_T(\alpha)$* energy signal with autocorrelation $k_{xx,T}(\tau) = \int_{-\infty}^{\infty} x_T(\alpha)x_T(\alpha+\tau)d\alpha$

Therefore:

$$K_{xx}(\tau)) = \lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}$$

Signal Auto-Correlation Function and Power Spectrum

$$K_{xx}(\tau)) = \lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}$$

An alternative definition of signal Power Spectrum is

$$S_x = F[K_{xx})]$$

The two definitions are consistent

$$S_x(f) = F[K_{xx}(\tau)] = F[\lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}] = \lim_{T\to\infty} \frac{F[k_{xx,T}(\tau)]}{2T} = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T}$$

and

$$P = K_{xx}(0) = \int_{-\infty}^{\infty} S_x(f) df$$

Signal Cross-Correlation Function (Power-type)

$$K_{xy}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)y(\alpha+\tau)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{x_T(\alpha)y_T(\alpha+\tau)}{2T} d\alpha$$

- x(t) and y(t) are two different signals, both power-type
- $K_{xy}(\tau)$ measures the degree of similarity of *x(t)* with *y(t)* shifted by τ to left (towards earlier time)
- If even only one of the two signals x(t) and y(t) is energy-type the energy type autocorrelation $k_{xy}(\tau)$ must be employed

  (in fact, the power-type crosscorrelation vanishes $K_{xy}(\tau) = 0$ and the energy-type crosscorrelation $k_{xy}(\tau)$ is finite).

Signal Auto-Correlation Function (Power-type)

- K_{xx} is symmetrical: $K_{xx}(\tau) = K_{xx}(-\tau)$
- K_{xx} has positive maximum at zero shift: $K_{xx}(0) > |K_{xx}(\tau)|$ with $K_{xx}(0) > 0$

Signal Cross-Correlation Function (Power-type)

- x(t) and y(t) are two different signals, both power-type
- K_{xy} is NOT symmetrical, however $K_{xy}(\tau) = K_{yx}(-\tau)$
- the maximum of K_{xy} is neither necessarily positive, nor at zero shift, however, the absolute maximum value is limited

$\left|K_{xy}(\tau)\right| \leq \frac{1}{2}\,[K_{xx}(0) + K_{yy}(0)]$ (linear mean of the maxima of k_{xx} and k_{yy})

$\left|K_{xy}(\tau)\right| \leq \sqrt{K_{xx}(0)\,K_{yy}(0)}$ (geometric mean of the maxima of k_{xx} and k_{yy})

Energy-signals and power-signals compared

Energy-type (pulses etc.)

Energy $E = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha$

Autocorrelation

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

Energy spectrum

$$S_{x,e} = F[k_{xx}(\tau)] = |X(f)|^2$$

and

$$\int_{-\infty}^{\infty} S_{x,e}(f)df = E$$

Power-type (periodic waveforms etc.)

Power $P = \lim_{T\to\infty} \int_{-T}^{T} \frac{x^2(\alpha)}{2T} d\alpha$

Autocorrelation

$$K_{xx}(\tau)) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha$$

Power spectrum

$$S_{x,p} = F[K_{xx}(\tau)] = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T}$$

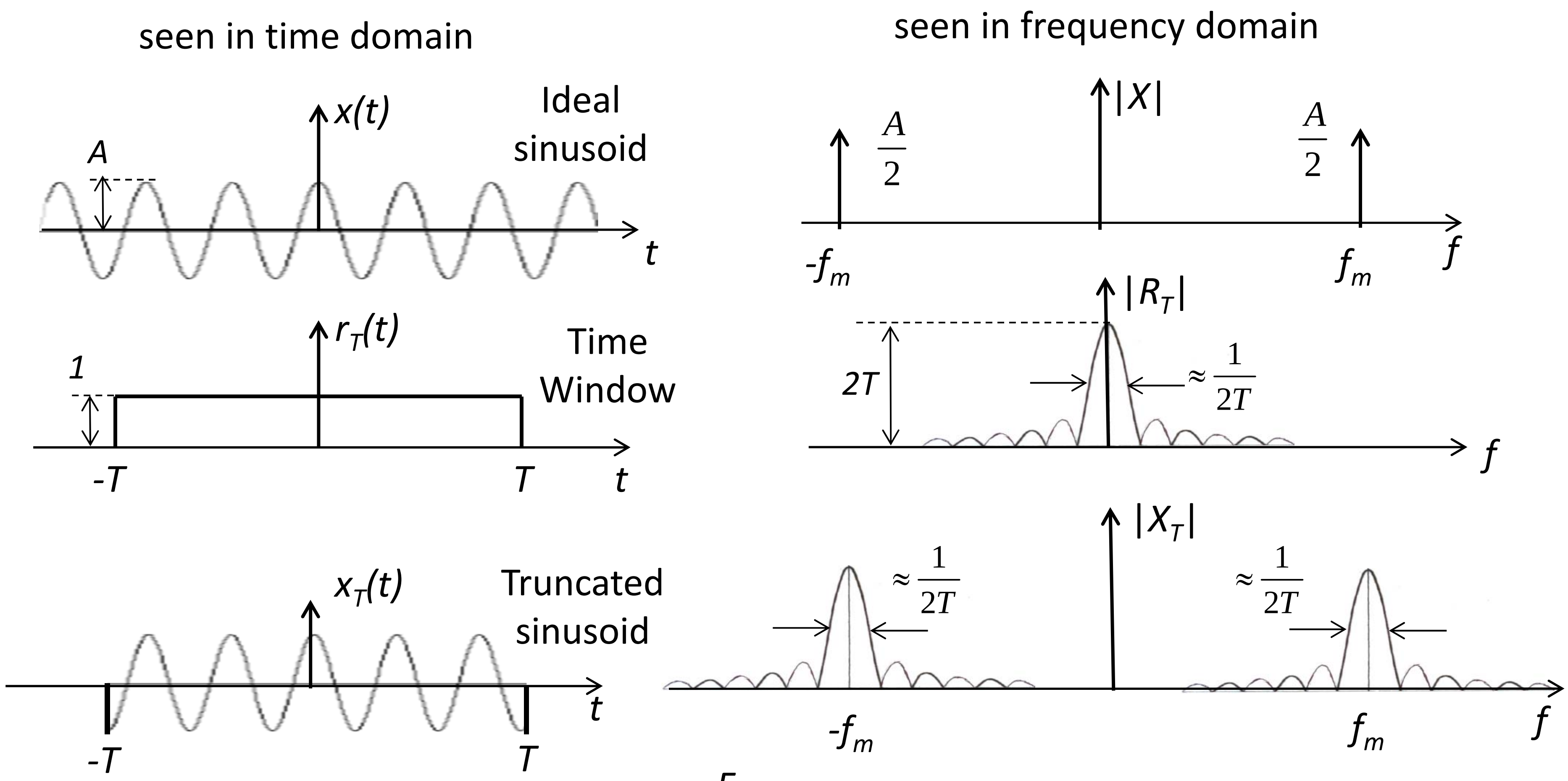
and

$$\int_{-\infty}^{\infty} S_{x,p}(f)df = P$$

Note on truncated signals

Note on truncated signals

Noteworthy case: truncated sinusoidal signal



$$x_T(t) = x(t) \cdot r_T(t) \quad \overset{F}{\underset{F^{-1}}{\rightleftarrows}} \quad X_T(f) = X(f) * R_T(f)$$

Note on truncated signals

- In reality the signal is always available over a finite time interval: therefore, in reality we always deal with truncated signals
- cropping in time corresponds to convolution of the signal in the *f* domain with the transform of the rectangle (*sinc* function)
- the convolution spreads the signal in the *f* domain; that is, it makes it wider and smoother
- the narrower is the window *2T*, the wider is the *sinc* and more significant is the signal spreading in frequency
- Applying correctly the sampling theorem we see that:
  the **sampling frequency** f_S to be employed for a **truncated** sinusoid of frequency f_m
  is **NOT** $f_S \approx 2f_m$; it must be REMARKABLY HIGHER $f_S >> 2f_m$
- NB1: in general the convolution of complex functions is difficult to visualize because
  a) it is twofold; it implies shifting in positive and in negative sense of the *f* axis;
  b) at every frequency *f* a sum of complex terms must be computed
- NB2: however, the case here considered is much simpler: at every frequency *f* only one contribution is significant, there is no sum to be computed.

APPENDIX: Crosscorrelation and Convolution

(for those who trust only analytical demonstrations)

$$k_{xy}(\tau) = \int_{-\infty}^{\infty} x(\beta)y(\beta+\tau)d\beta$$

change of variable :

$$\beta = -\alpha$$
$$\mathrm{d}\beta = -d\alpha$$

shows that

$$k_{xy}(\tau) = \int_{-\infty}^{\infty} x(\beta)y(\beta+\tau)d\beta = -\int_{+\infty}^{-\infty} x(-\alpha)y(-\alpha+\tau)d\alpha =$$
$$= \int_{-\infty}^{\infty} x(-\alpha)y(\tau-\alpha)d\alpha$$

that is

$$k_{xy}(\tau) = x(-\alpha) * y(\alpha)$$