Sensors, Signals and Noise

COURSE OUTLINE

- Introduction
- Signals and Noise: 1) Description
- Filtering
- Sensors and associated electronics

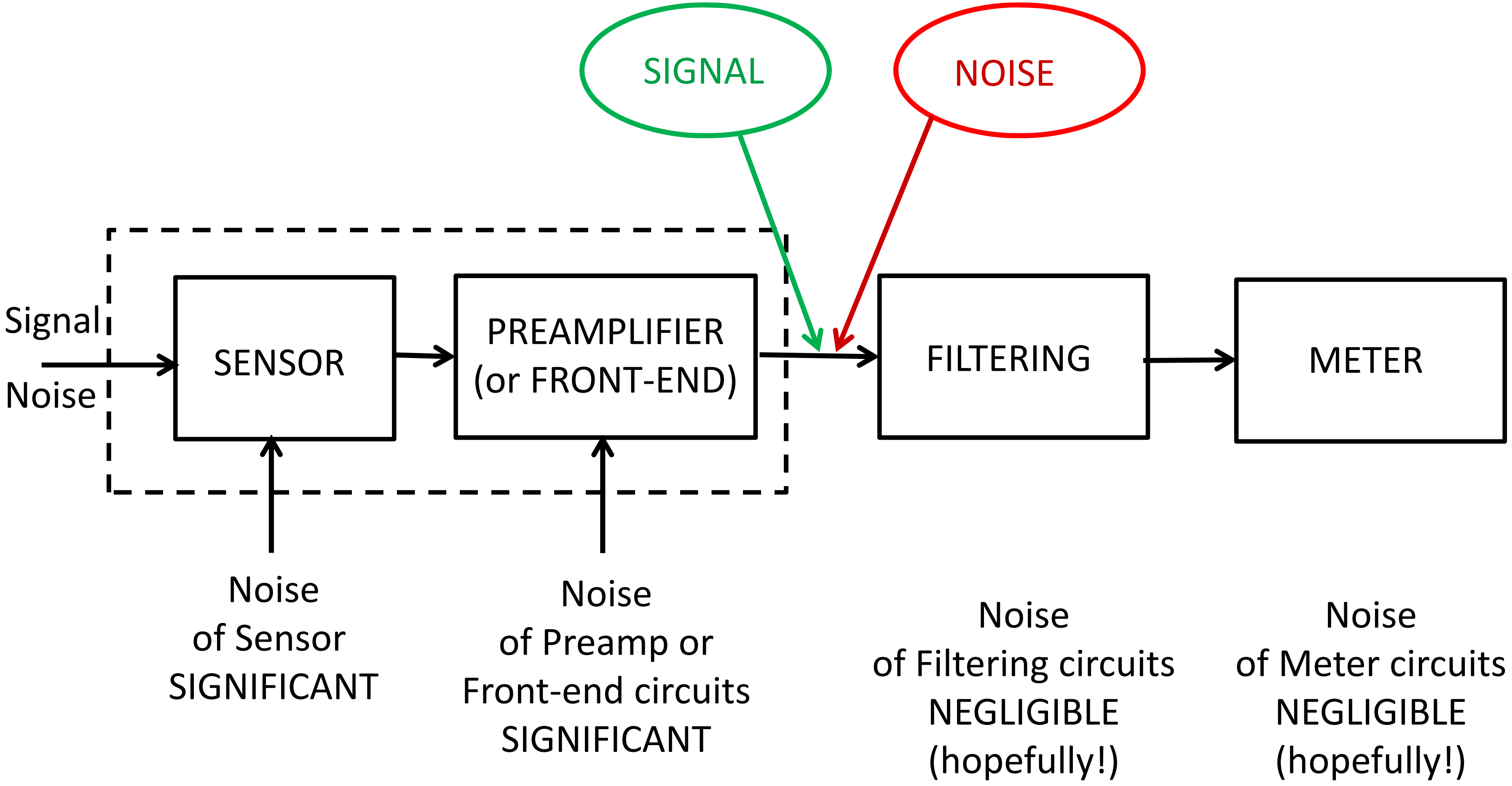
Noise Description

- Noise Waveforms and Samples
- Statistics of Noise Samples and Probability Distribution (PD)
- Complete Description of Noise with Probability Distributions
- Basic Description of Noise with the 2°order Moments of PD
- Autocorrelation Function of Noise
- Power Spectrum of Noise

and for those who trust only analytical demonstrations

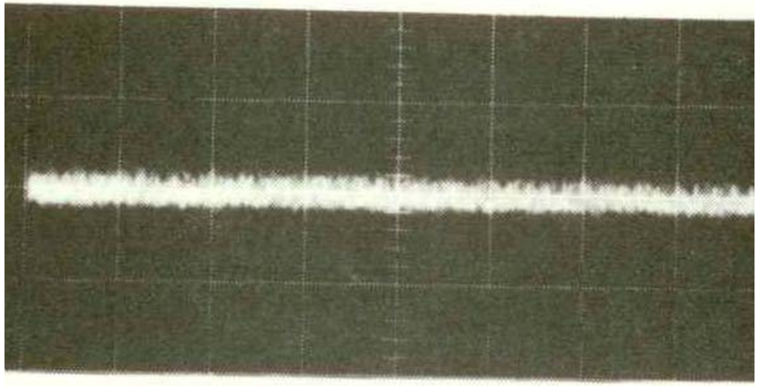
- APPENDIX: Exchanging the order of Time-Averaging and Ensemble-Averaging in the definition of Power Spectrum

Set-Up of Sensor Measurements



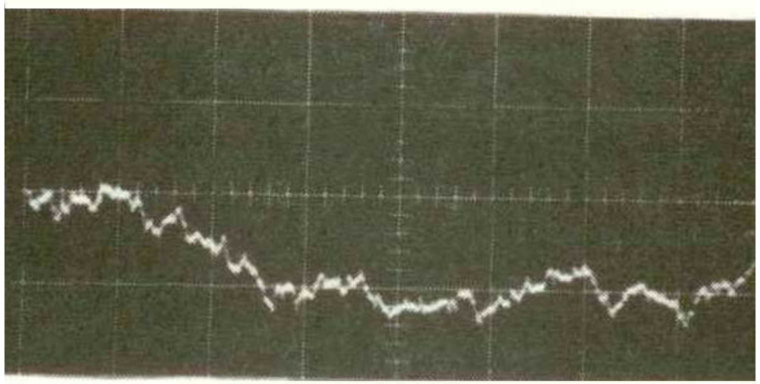
Noise Waveforms and Samples

Noise waveforms (oscilloscope @ 50μs/div)



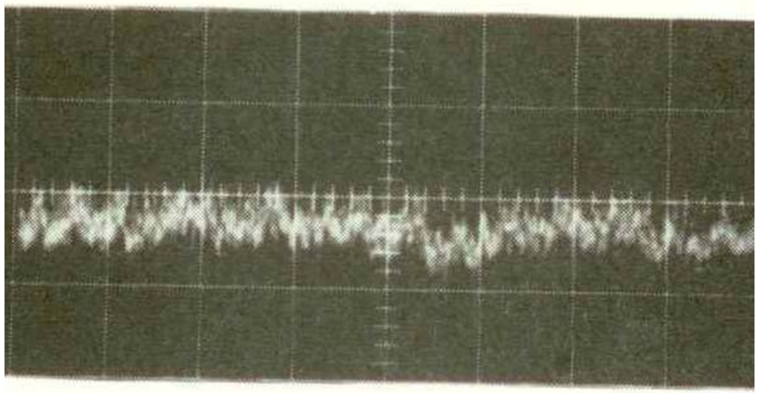
White Noise

spectrum ***S = constant***



Random-Walk Noise

spectrum $\boldsymbol{S} = \frac{\mathbf{1}}{\boldsymbol{f}^{\mathbf{2}}}$

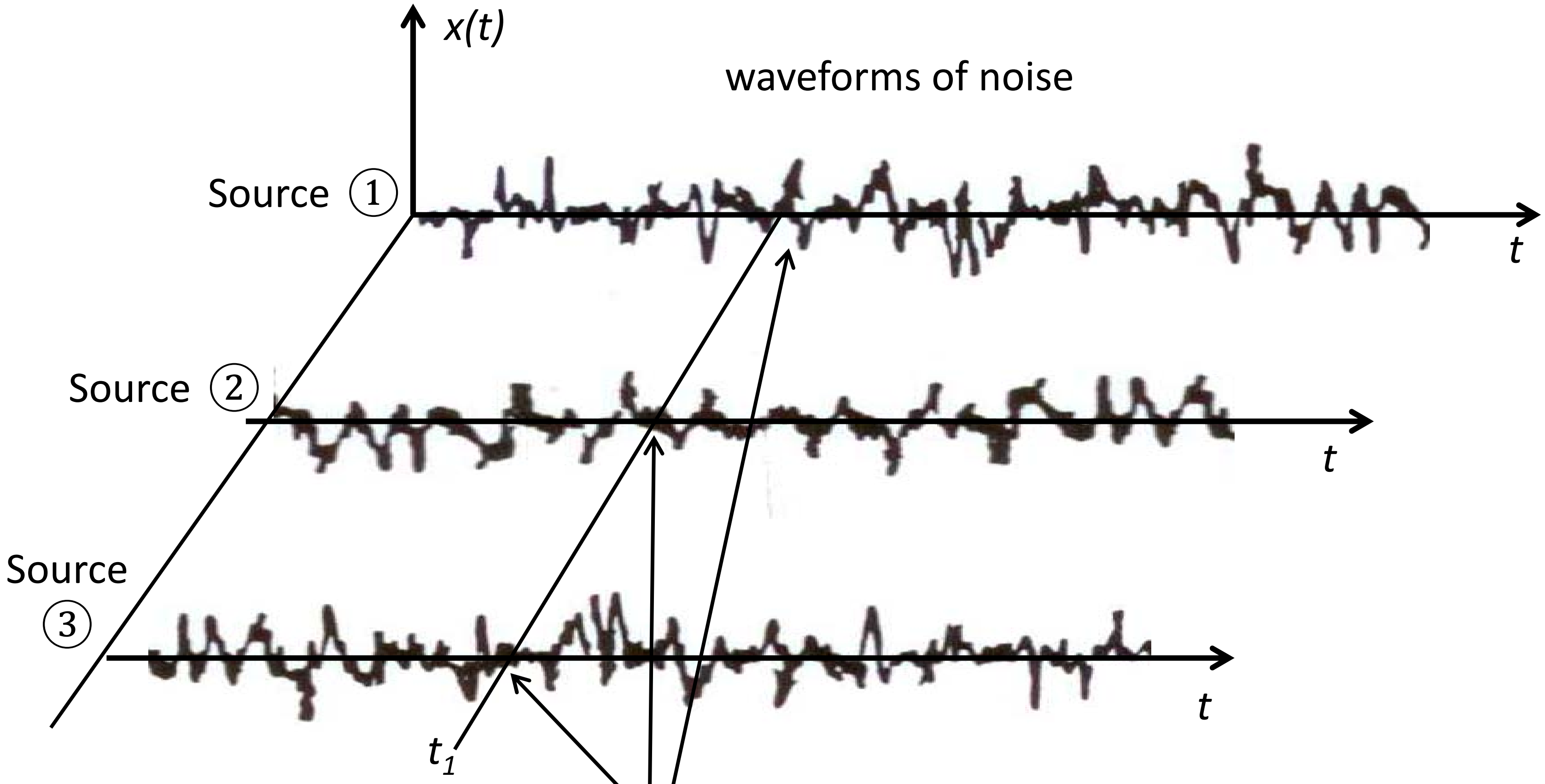


Flicker Noise

spectrum $\boldsymbol{S} = \frac{\mathbf{1}}{\boldsymbol{f}}$

Noise Waveform Ensemble

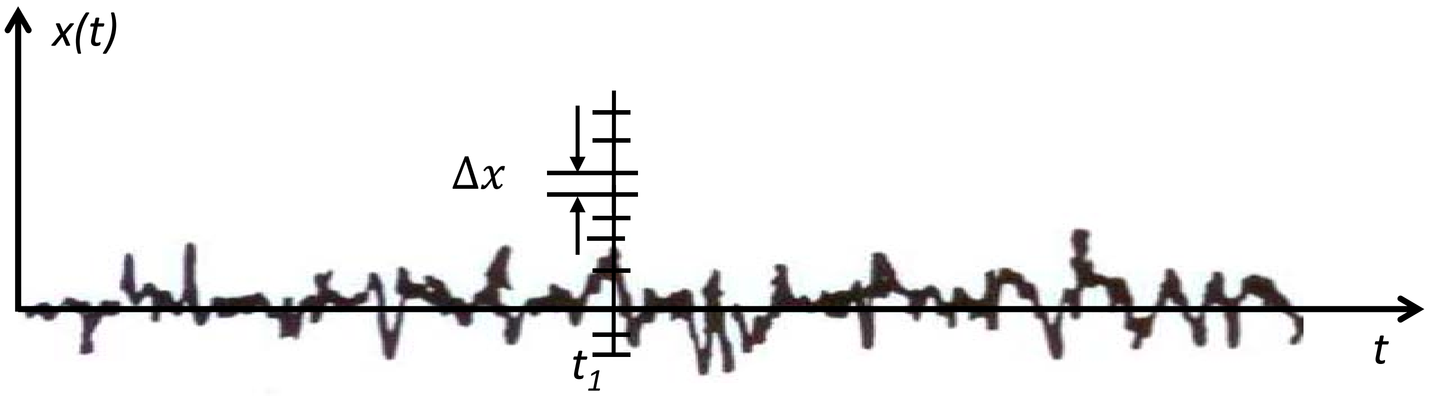
Set of identical noise sources (many **identical** amplifiers or resistors or other)



$x(t_1)$ amplitude sample at time t_1 on each waveform

Statistics of Noise Samples
and
Probability Distribution (PD)

Classifying the Amplitude of Noise Samples



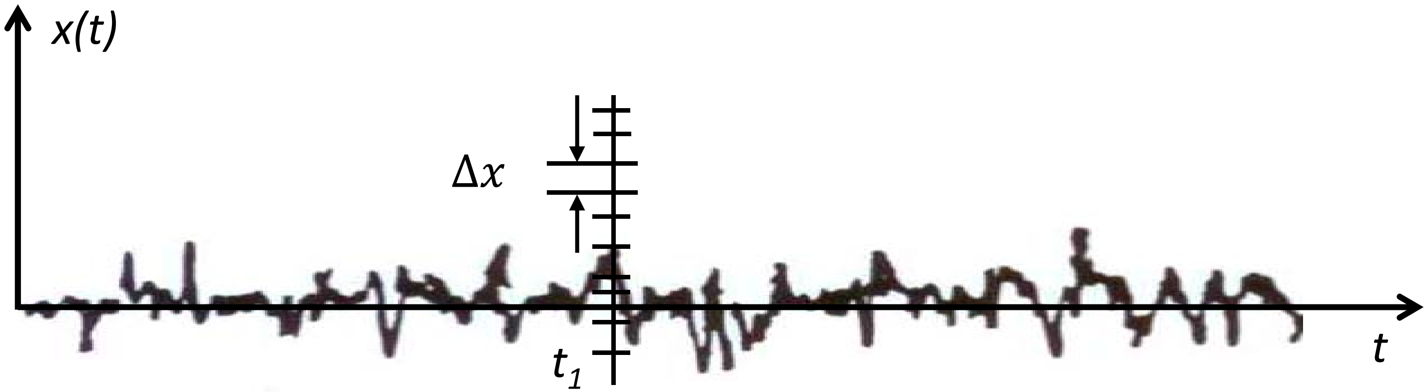
The amplitude $x(t_1)$ of the noise waveform at time t_1
is compared to a scale of discrete values x_k spaced by constant interval Δx
and is classified at the nearest value x_k of the scale

A high number N of noise waveform is sampled and measured

of which ΔN_k is the number of sample waveforms classified at x_k

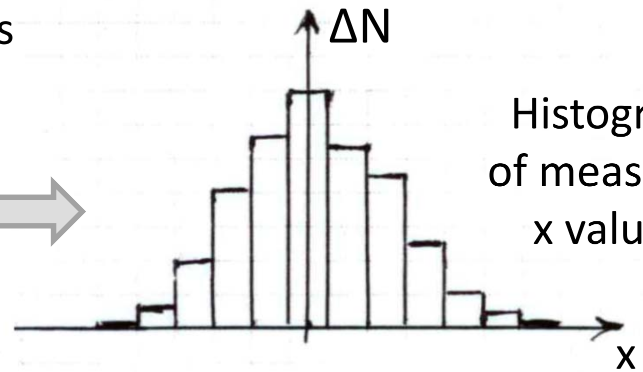
$\Delta f_k = \frac{\Delta N_k}{N}$ is called <u>statistical frequency</u> of the amplitude x_k

Noise Sample Statistics and Probability



N values $x(t_1)$ measured (in units Δx) in N waveforms

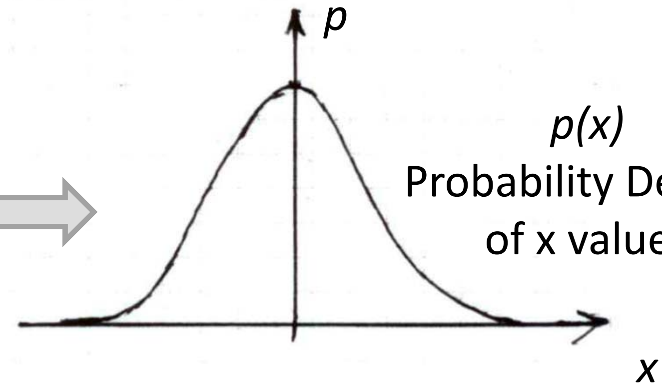
ΔN_0 in the central Δx (around x=0)
ΔN_1 in the first Δx (centered in $x_1 = \Delta x$)
....
ΔN_k in the k-th Δx (centered in $x_k = k\Delta x$)



Histogram of measured x values

statistical frequency of x_k is $\Delta f_k = \frac{\Delta N_k}{N}$

- if $\Delta x \rightarrow dx$ then $\Delta N_k \rightarrow dN_k = n(x_k)dx$
- hence $\Delta f_k \rightarrow df_k = \frac{dN_k}{N} = \frac{n(x_k)}{N} dx$
- if $N \rightarrow \infty$ then $df_k = \frac{n(x_k)}{N} dx = p(x)dx$



p(x)
Probability Density of x values

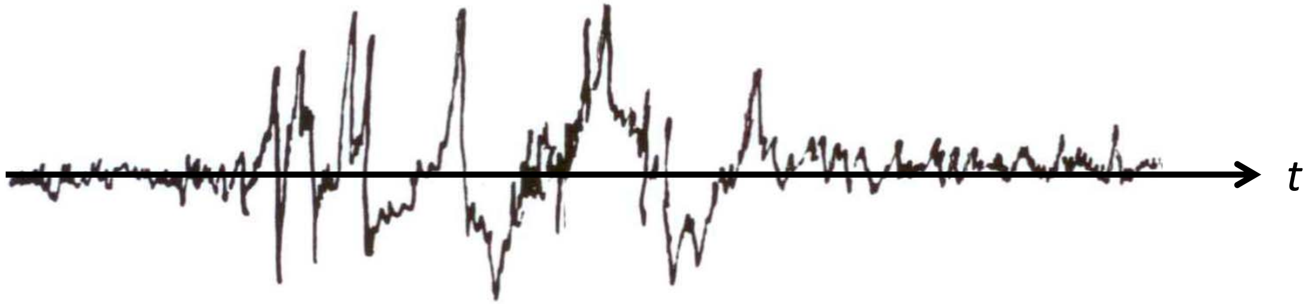
Stationary and Non-stationary Noise

STATIONARY noise :

the **probability density is constant** in time $p = p(x)$

NON-STATIONARY noise :

the **probability density varies** in time $p= p(x, t)$

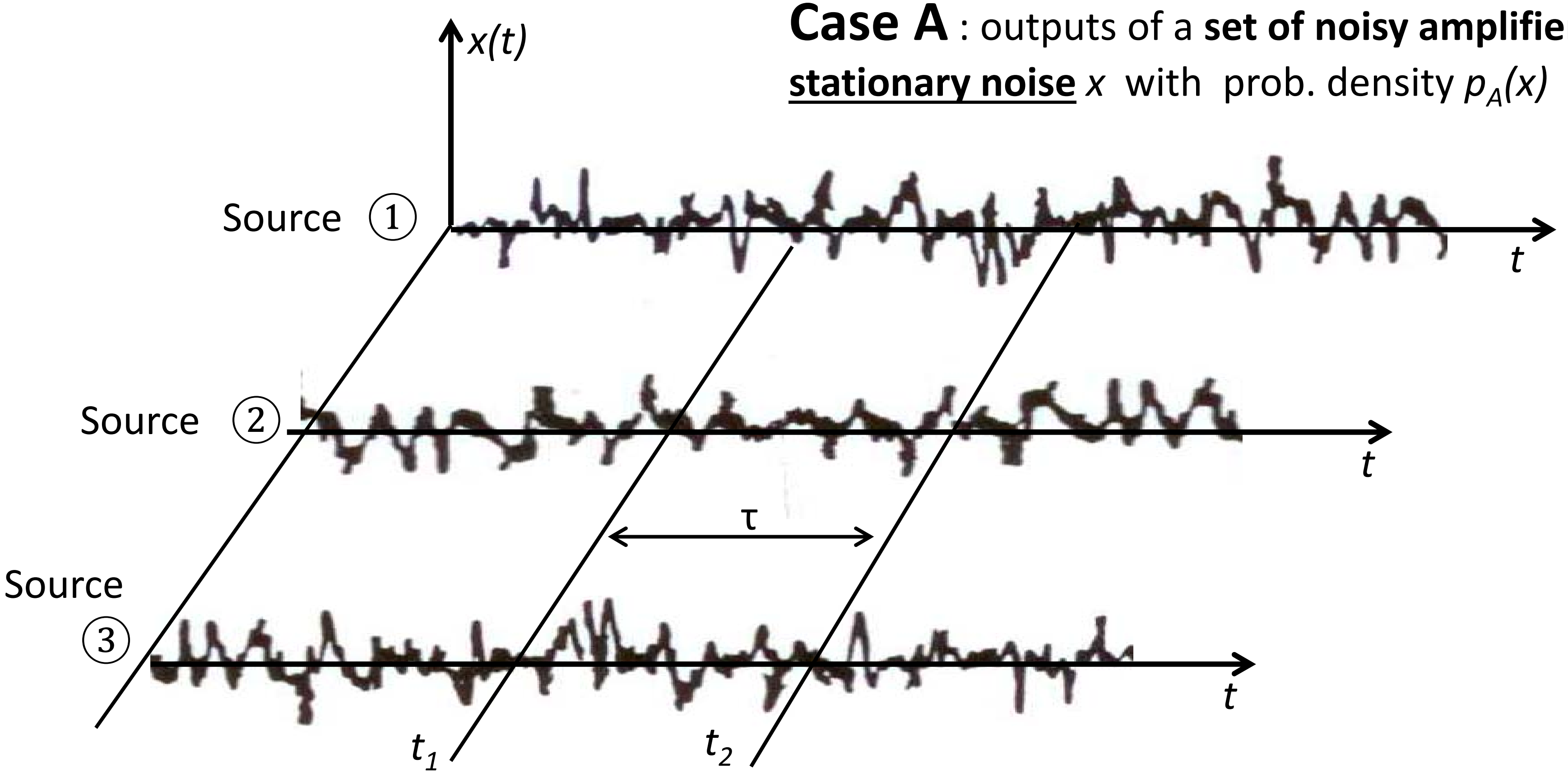


BEWARE!!

the probability density ***p* alone does not** give a complete description of the noise, in fact <u>different cases</u> can have <u>equal probability</u> density *p*

Noise Waveforms and Sample Statistics

Case A : outputs of a **set of noisy amplifiers**, **stationary noise** x with prob. density $p_A(x)$

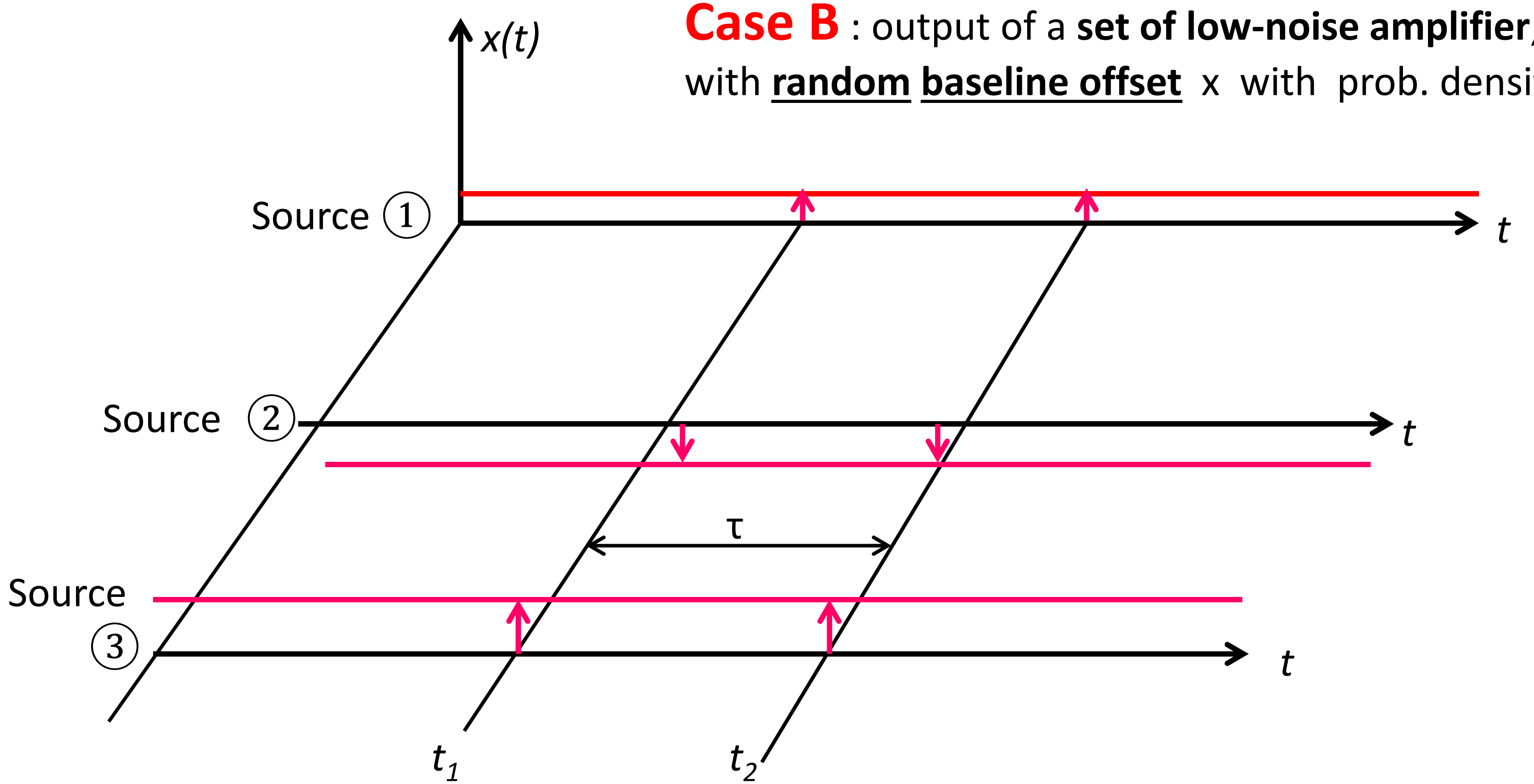


Values $x(t_1)$ and $x(t_2)$ measured on a sample waveform at different t_1 and t_2 are random values with equal probability density $p_A(x)$ and they are

- in practice identical for ultra-short interval τ
- somewhat different for short interval τ
- different and independent for longer interval τ

Noise Waveforms and Sample Statistics

Case B : output of a **set of low-noise amplifier**,
with **random baseline offset** x with prob. density $p_B(x)$



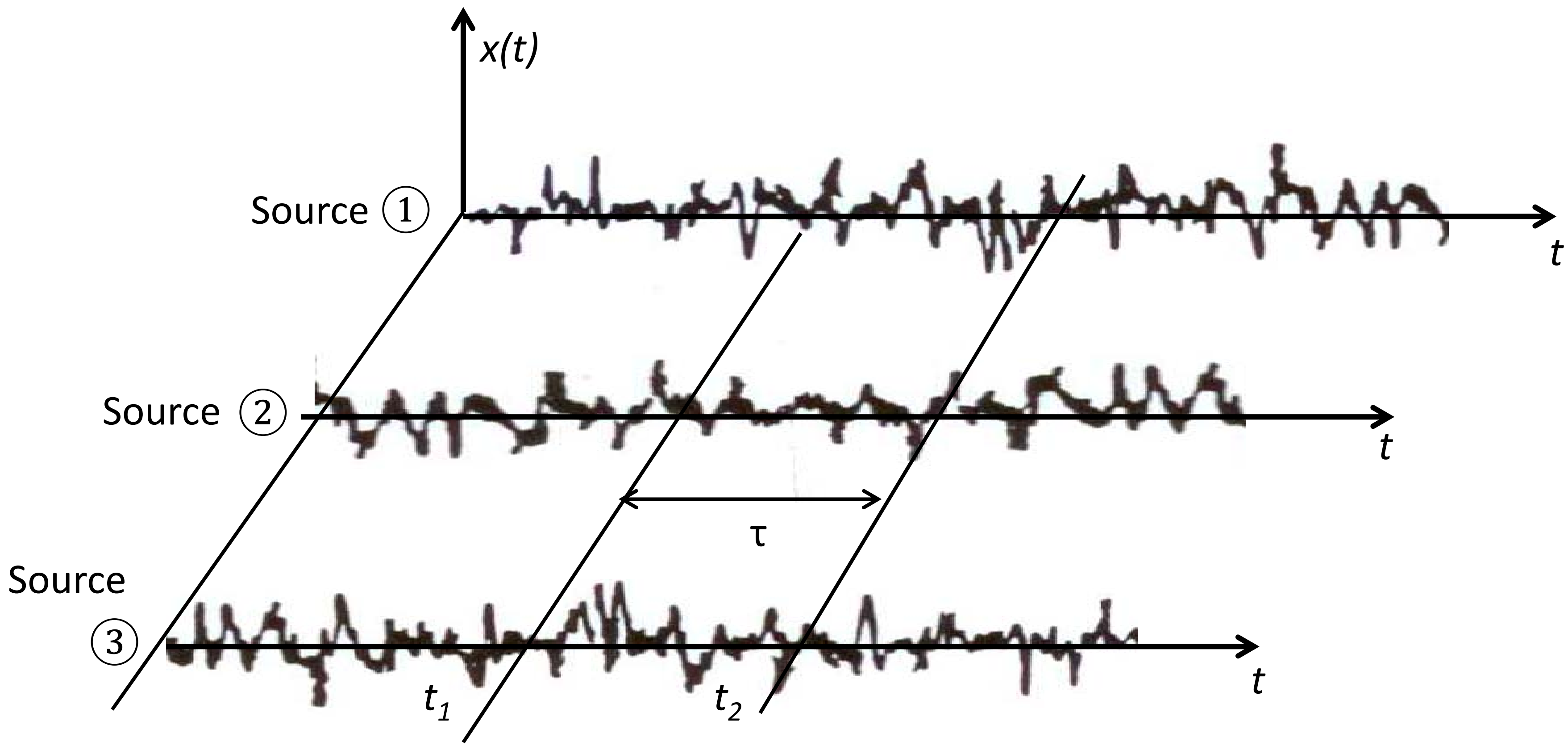
Values $x(t_1)$ and $x(t_2)$ measured on a sample waveform at different t_1 and t_2 :

- they are random values with probability density $p_B(x)$;
- they are equal for any interval τ , short or long

Case B is **different** from A, but it can have **equal probability density** $p_B(x) = p_A(x)$

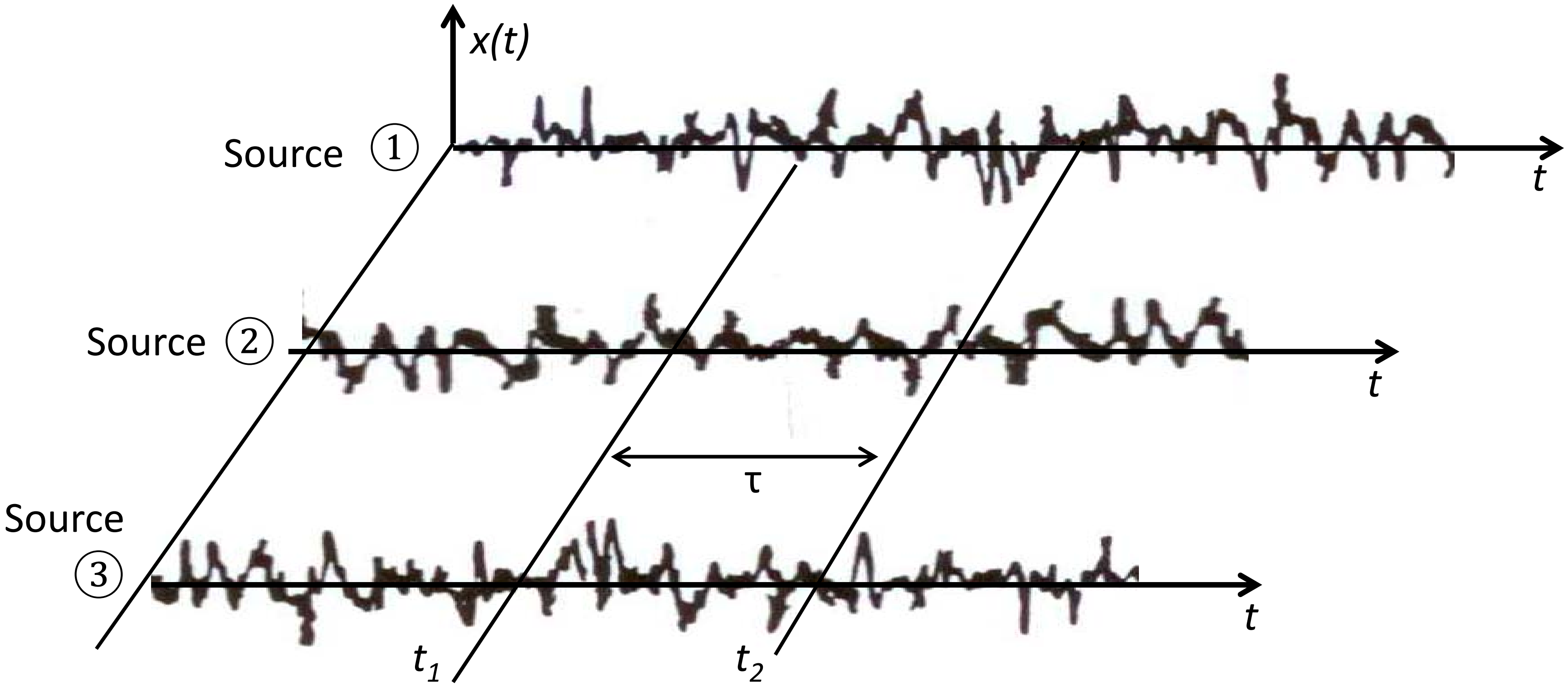
Complete Description of Noise with Probability Distributions

Full Description of Noise



- For a proper description of the noise the <u>marginal</u> probability $\boldsymbol{p_m(x, t)dx}$ of having a value x at time t is <u>NOT sufficient</u>
- The <u>joint</u> probability $\boldsymbol{p_j(x_1, x_2, t_1, t_2)dx_1\, dx_2}$ of having a value x_1 at time t_1 **and** a value x_2 at time t_2 <u>must also be considered</u>

Noise Description with Probability Distributions



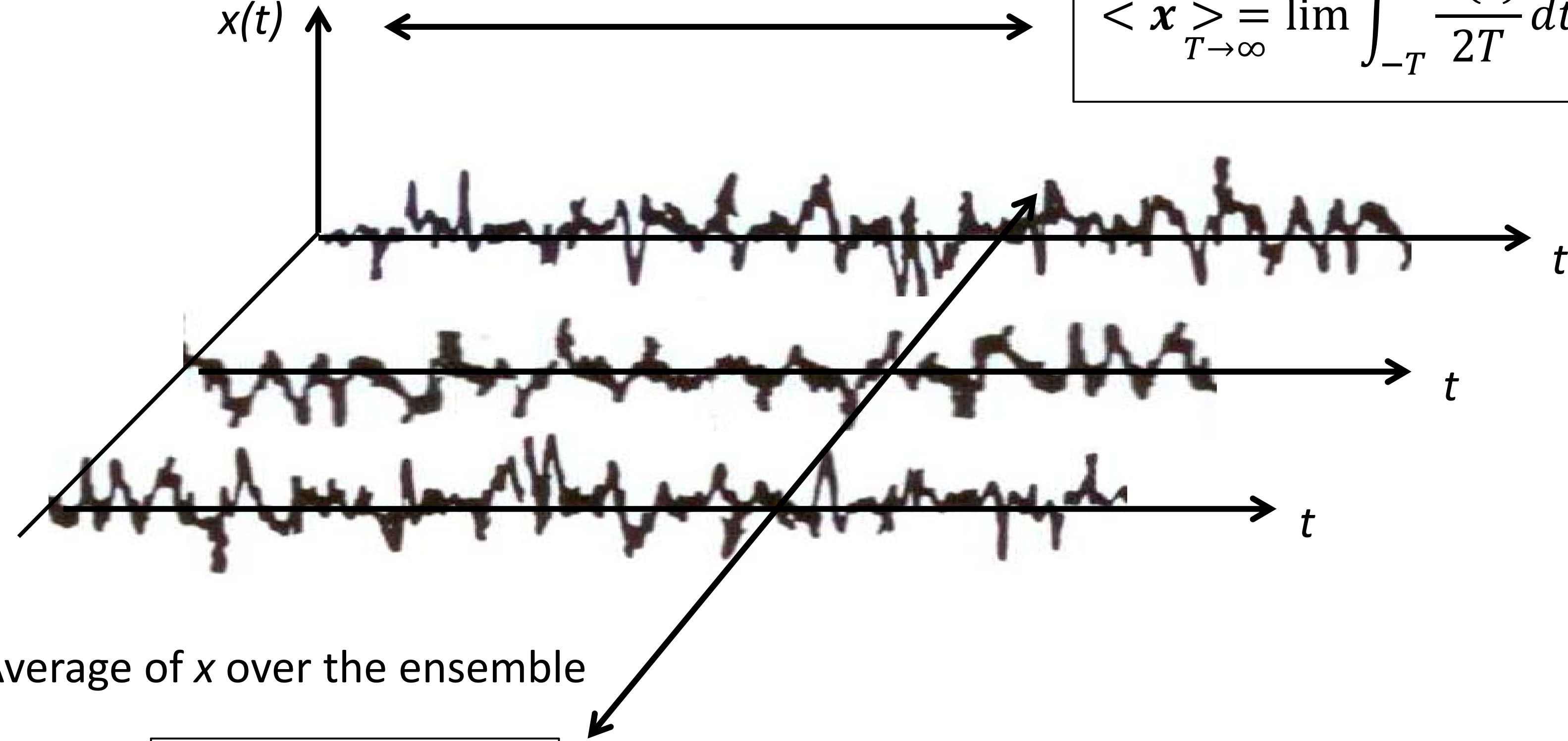
A full description of the noise is obtained by knowing:

- The **marginal** probability density $p_m(x) = p_m(x; t_1)$ for **every** instant $\boldsymbol{t_1}$.
  For stationary noise p_m does NOT depend on time t_1 : $p_m = p_m(x)$
- The **joint** probability density $p_j (x_1 , x_2) = p_j(x_1 , x_2 ; t_1, t_2) = p_j(x_1 , x_2 ; t_1, t_1 + \tau)$
  for **every couple** of instants $\boldsymbol{t_1}$ and $\boldsymbol{t_2 = t_1 + \tau}$.
  For stationary noise $\boldsymbol{p_j}$ depends only on the time interval $\boldsymbol{\tau}$, NOT on the time position $\boldsymbol{t_1}$

Note: Time-Average and Ensemble-Average

$< \boldsymbol{x} >$ Average of *x* over the time

$$< \boldsymbol{x} > = \lim_{T \to \infty} \int_{-T}^{T} \frac{x(t)}{2T} dt$$



$\overline{\boldsymbol{x}}$ Average of *x* over the ensemble

$$\overline{\boldsymbol{x}} = \int_{-\infty}^{\infty} x \, p(x) dx$$

Basic Description of Noise with 2nd order Moments of Probability Distribution

NOTE: Moments of Probability Distributions

NB: for clarity, we call here the two statistical variables x and y instead of x_1 and x_2

Moments of a marginal *p(x)* $m_n = \overline{x^n} = \int_{-\infty}^{\infty} x^n \, p(x) dx$

Moments of a joint *p(x,y)* $m_{jk} = \overline{x^j \, y^k} = \int_{-\infty}^{\infty} x^j \, y^k p(x,y) dx dy$

- the m_n (and m_{jk}) give information on the features of the distributions
- as the order (*n* or *j+k*) increases, the information is increasingly of detail

Let's consider a description of noise limited to the 2° order moments, i.e.

Mean square value (or variance)

$$m_2 = \overline{x^2} = \int_{-\infty}^{\infty} x^2 \, p(x) dx = \sigma_x{}^2$$

Mean product value (or covariance of *x* and *y*)

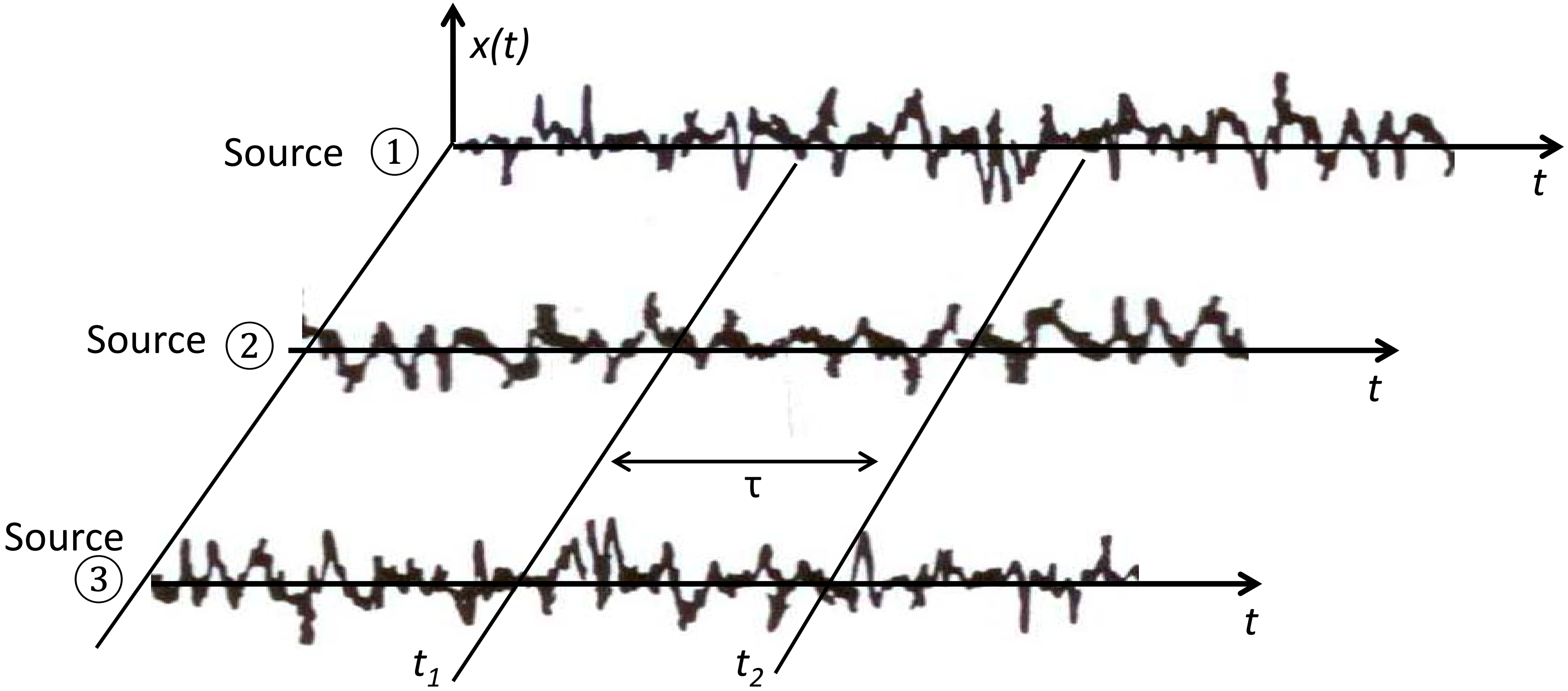
$$m_{11} = \overline{xy} = \int_{-\infty}^{\infty} xy \, p(x,y) dx dy = \sigma_{xy}{}^2$$

NB: it is obviously

$m_o = m_{oo} = 1$ the total probability is normalized to 1

$m_1 = m_{1o} = \bar{x} = 0 = \bar{y} = m_{01}$ the mean value of noise is zero

Noise Description with 2°order Moments



- for <u>every</u> instant t_1 the <u>mean square value</u> (or variance) $\overline{x^2(t_1)} = \sigma_x^2(t_1)$
  For stationary noise $\overline{x^2}$ does NOT depend on time t_1
- for <u>every couple</u> t_1 and $t_2 = t_1 + \tau$ the <u>meanproduct</u> $\overline{x(t_1)x(t_2)} = \overline{x(t_1)x(t_1+\tau)}$
  For <u>stationary</u> noise it depends only on the <u>time interval</u> τ, NOT on the time position t_1

Autocorrelation Function of Noise

Noise Description with the Autocorrelation Function

$$\overline{x\,(t_1)x\,(t_2)} = \overline{x\,(t_1)x\,(t_1+\tau)} = R_{xx}(t_1, t_1+\tau) = R_{xx}(t_1, t_2)$$

- is called **Autocorrelation Function** of the noise
- is <u>always</u> a function of the <u>interval</u> τ between the two instants t_1 and t_2
- is also a function of t_1 only for <u>non-stationary</u> noise

NOTE THAT:

for a noise *x* the autocorrelation $R_{xx}(\tau)$ is an <u>ensemble-average</u>,

for a signal *x* the autocorrelation function $K_{xx}(\tau)$ is a <u>time-average</u>

The **noise mean square value** is called **NOISE POWER**

it is the autocorrelation with τ = 0

$$\overline{x^2(t)} = R_{xx}(t, 0)$$

for <u>stationary</u> noise it is <u>constant</u> at any *t*

$$\overline{x^2} = R_{xx}(0)$$

Power Spectrum of Noise

Noise Description with the Power Specrum

Noise has power-type waveforms (divergent energy → ∞)

which have statistical variations from waveform to waveform of the ensemble.

By **averaging over the ensemble** of the autocorrelations of the noise waveforms ,

the concepts of power and power spectrum introduced **<u>for the signals</u>**

can be **extended to the noise**

$$P = \overline{\lim_{T\to\infty} \int_{-T}^{T} \frac{x^2(\alpha)}{2T} d\alpha} = \overline{\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{x_T^{\,2}(\alpha)}{2T} d\alpha} = \overline{\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{\left|X_T(f)\right|^2}{2T} df} =$$

$$= \int_{-\infty}^{\infty} \overline{\lim_{T\to\infty} \frac{\left|X_T(f)\right|^2}{2T}} df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{\overline{\left|X_T(f)\right|^2}}{2T} df$$

Therefore, the Power Spectrum of the noise is defined as

$$S_x(f) = \lim_{T\to\infty} \frac{\overline{\left|X_T(f)\right|^2}}{2T}$$

and the noise power is

$$P = \int_{-\infty}^{\infty} S_x(f) df$$

Noise Description with the Power Spectrum

By averaging over the ensemble we can extend to the noise

also the second definition of Power Spectrum introduced for the signals

$$S_x(f) = \overline{F[K_{xx}(\tau)]} = F[\ \overline{K_{xx}(\tau)}\ ] =$$

$$= F[\overline{\lim_{T\to\infty} \frac{\int_{-\infty}^{\infty} x_T(\alpha)x_T(\alpha+\tau)d\alpha}{2T}}] =$$

$$= \mathrm{F}[\overline{\lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}}] = \lim_{T\to\infty} \frac{F[\overline{k_{xx,T}(\tau)}]}{2T}$$

The Power Spectrum of the noise can be directly defined as

$$S_x(f) = \lim_{T\to\infty} \frac{\overline{|X_T(f)|^2}}{2T}$$

The noise power is

$$P = \int_{-\infty}^{\infty} S_x(f)df = \overline{K_{xx}(0)}$$

Bilateral and Unilateral Spectral Power Density

- The mathematical spectral density $S_x(f)$ defined over $-\infty < f < \infty$,

  is a <u>bilateral</u> spectral density $S_{xB}(f)$

  attention is called on this fact by the second subscript B

- The noise power computed <u>with the bilateral density</u> S_{xB} is

$$P = \int_{-\infty}^{\infty} S_{xB}(f)df$$

- Since $S_{xB}(f)$ is symmetrical $S_{xB}(-f) = S_{xB}(+f)$, it is

$$P = 2 \int_0^{\infty} S_{xB}(f)df = \int_0^{\infty} 2S_{xB}(f)df$$

- A <u>unilateral</u> «physical» spectral density $S_{xU}(f) = 2S_{xB}(f)$ is usually employed in engineering tasks for making computations only in the positive frequency range

- The noise power computed with <u>with the unilateral density</u> S_{xU} is

$$P = \int_0^{\infty} S_{xU}(f)df$$

Power Spectrum of Non-Stationary Noise

$$S_x(f) = F[\,\overline{K_{xx}(\tau)}\,]$$

$\overline{K_{x\,x}(\tau)}$ results from the double average,

first over the time $K_{xx}(\tau) =< x(t)x(t+\tau) >$ then over the ensemble

It can be shown that the order of averaging can be exchanged *(see later)*

$$\overline{K_{x\,x}(\tau)} = \overline{< x(t)x(t+\tau) >} =< \overline{x(t)x(t+\tau)} > = < R_{xx}(t, t+\tau) >$$

The power spectrum thus is related to the ensemble autocorrelation function

$$S_x(f) = F[< R_{xx}(t, t+\tau) >]$$

- For **non-stationary noise** $S_x(f)$ can be defined with reference to the **time-average of the ensemble autocorrelation** function of the noise.

- For **stationary** noise there is no need of time-averaging: it is simply

$$< R_{xx}(t, t+\tau) > = R_{xx}(\tau)$$

and

$$S_x(f) = F[R_{xx}(\tau)]$$

APPENDIX :

the order of Time-Averaging and Ensemble-Averaging can be exchanged in the definition of the Noise Power Spectrum

Let's verify that $\overline{K_{x\,x}(\tau)} = < R_{xx}(t, t+\tau) >$

In fact:

$$\overline{K_{x\,x}(\tau)} = \overline{\lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha} =$$

$$= \lim_{T\to\infty} \int_{-T}^{T} \frac{\overline{x(\alpha)x(\alpha+\tau)}}{2T} d\alpha$$

$$= \lim_{T\to\infty} \int_{-T}^{T} \frac{R_{xx}\,(\alpha, \alpha+\tau)}{2T} d\alpha =$$

$$= < R_{xx}(t, t+\tau) >$$