Sensors, Signals and Noise

COURSE OUTLINE

- Introduction
- Signals and Noise: 3) Analysis and Simulation
- Filtering
- Sensors and associated electronics

Noise Analysis and Simulation

- White Noise
- Band-Limited White Noise (Wide-Band Noise)
- Basic Parameters of Wide-Band Noise
- Foundations of White-Noise Filtering
- Generation and Simulation of Any Noise by a Poisson Process

and for those who wish to gain a better insight

- APPENDIX: Noise Power Transients

White Noise

White Noise (stationary)

IDEAL «white» noise

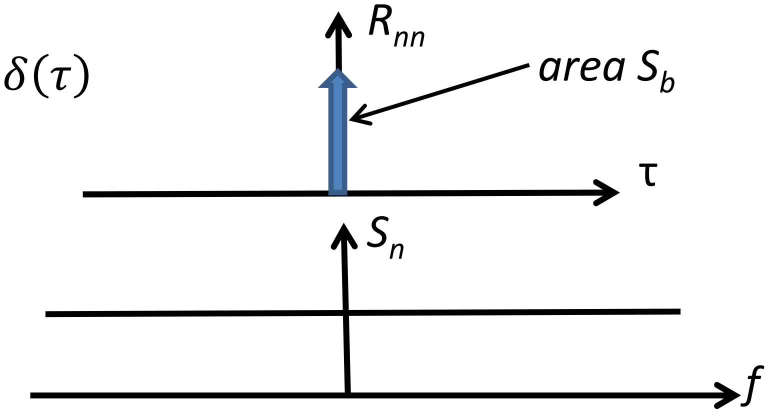
is a concept extrapolated from Johnson noise and shot noise defined by its **essential** feature:

no autocorrelation at any time distance τ, no matter how small

$$R_{nn}(\tau) = S_b \cdot \delta(\tau)$$

and therefore constant spectrum

$$S_n(f) = S_b$$



In reality such a noise does not exist: it would have divergent power $\overline{n^2} \to \infty$

REAL «white» noise has

- **Very small width** of autocorrelation, shorter than the minimum time interval of interest in the actual case and therefore approximated to zero
- **Very wide band** with constant spectral density S_b , wider than the maximum frequency of interest in the actual case and therefore approximated to infinite

White Noise (non-stationary)

Also in non-stationary cases the IDEAL «white» noise
is defined by the **essential** characteristic feature:

no correlation at any finite time distance *τ*, no matter how small,

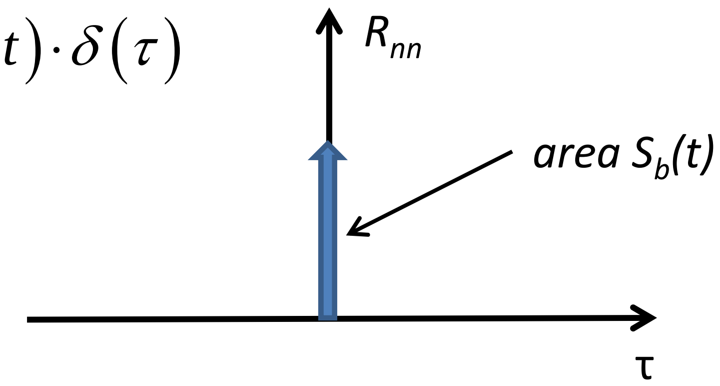
but the noise intensity is no more constant, it varies with time t

that is

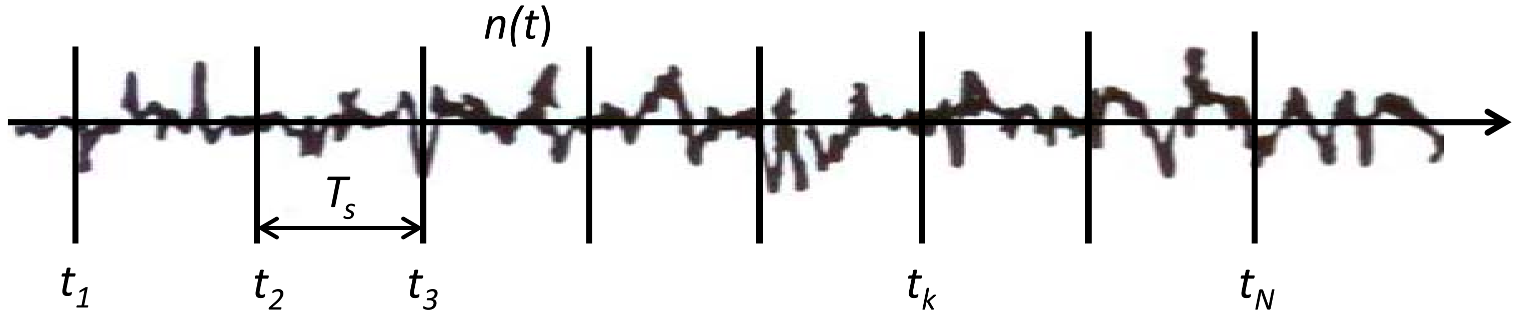
the autocorrelation function is δ-like,

but has time-dependent area $S_b(t)$

$$R_{nn}(t, t+\tau) = S_b(t) \cdot \delta(\tau)$$



Filtering white noise is simple



For clarity, let's consider a discrete case:

linear filtering in digital signal processing:

- Sample n_1 at t_1 and multiply by a weight w_1 ,
- sample n_2 at $t_2 = t_1 + T_s$ and multiply by a weight w_2 and sum
- and so on ....

The filtered noise n_f is

$$n_f = w_1 n_1 + w_2 n_2 + \ \ldots\ldots = \sum_{k=1}^{N} w_k n_k$$

and its mean square value is

$$\overline{n_f^{\,2}} = \overline{w_1^{\,2} n_1^{\,2} + w_2^{\,2} n_2^{\,2} + \ \ldots + w_1 n_1 \cdot w_2 n_2 + w_1 n_1 \cdot w_3 n_3 + \cdots} =$$

$$= w_1^{\,2}\overline{n_1^{\,2}} + w_2^{\,2}\overline{n_2^{\,2}} + \ \ldots\ w_1\, w_2 \overline{n_1 n_2} + w_1 w_3 \overline{n_1 n_3} + \cdots$$

Filtering white noise is simple

$$\overline{n_f^{\,2}} = \overline{w_1^{\,2} n_1^{\,2} + w_2^{\,2} n_2^{\,2} + \ \dots + w_1 n_1 \cdot w_2 n_2 + w_1 n_1 \cdot w_3 n_3 + \cdots} =$$

$$= w_1^{\,2}\overline{n_1^{\,2}} + w_2^{\,2}\overline{n_2^{\,2}} + \ \dots\ w_1\, w_2 \overline{n_1 n_2} + w_1 w_3 \overline{n_1 n_3} + \dots$$

If noise at interval T_s is **not correlated**, then all rectangular terms vanish

$$\overline{n_1 n_2} = \overline{n_1 n_2} = \cdots = 0$$

and the result is **simply a sum of squares**, even in case of non-stationary noise

$$\overline{n_f^{\,2}} = w_1^{\,2}\overline{n_1^{\,2}} + w_2^{\,2}\overline{n_2^{\,2}} + \ \dots =$$

$$= \sum_{k=1}^{N} w_k^{\,2}\overline{n_k^{\,2}}$$

If the noise is stationary

$$\overline{n_1^{\,2}} = \overline{n_2^{\,2}} = \overline{n_3^{\,2}} = \cdots = \overline{n^2}$$

there is a further simplification

$$\overline{n_f^{\,2}} = \overline{n^2}\,(w_1^{\,2} + w_2^{\,2} + \ \dots) =$$

$$= \overline{n^2}\ \sum_{k=1}^{N} w_k^{\,2}$$

we will see later that also with continuous filtering white noise brings similar simplification

Band-Limited White Noise

or

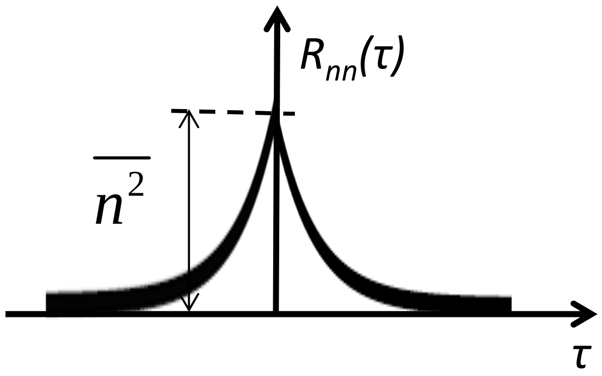
Wide-Band Noise

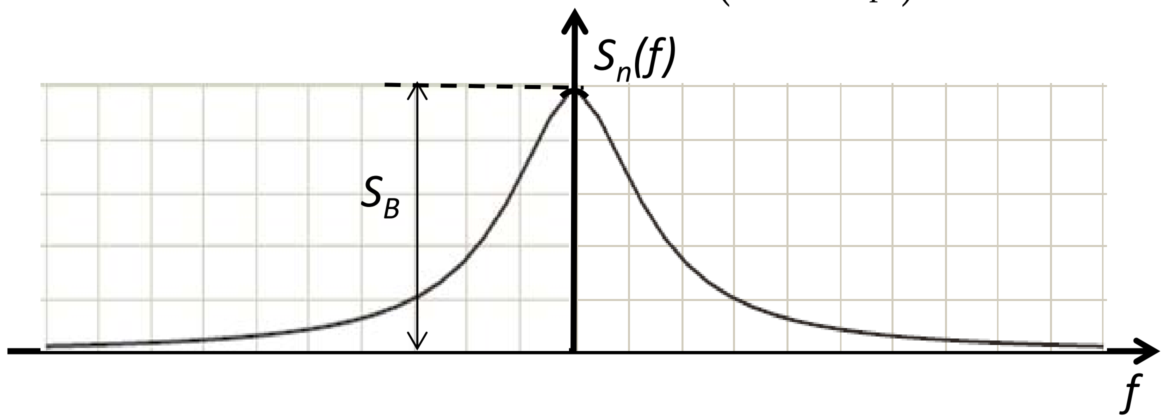
Band-limited white noise (wide-band noise)

- Real white noise = white noise with band limited at high frequency
- The limit may be inherent in the noise source or due to low-pass filtering enforced by the circuitry. Anyway, in all real cases there is such a limit
- A frequent typical case is the **Lorentzian** spectrum:
  band limited by a **simple pole** with time constant T_p , pole frequency $f_p = 1/ 2\pi T_p$

$$R_{nn}(\tau) = \overline{n^2} e^{-\frac{|\tau|}{T_p}}$$

$$S_n(f) = \frac{S_B}{1+\left(2\pi f T_p\right)^2}$$



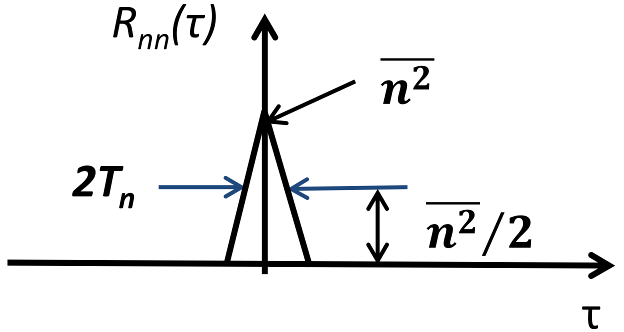


Basic Parameters of Wide-Band Noise

Simplified description of wide-band noise

The true $R_{nn}(\tau)$ and $S_n(f)$ can be **approximated** by simple functions **retaining the noise main features**:

a) equal **mean square** $\overline{\boldsymbol{n^2}}$ and b) equal spectral **density** $\boldsymbol{S_b}$



in time: $R_{nn}(\tau)$ triangular approx, half-width $2T_n$

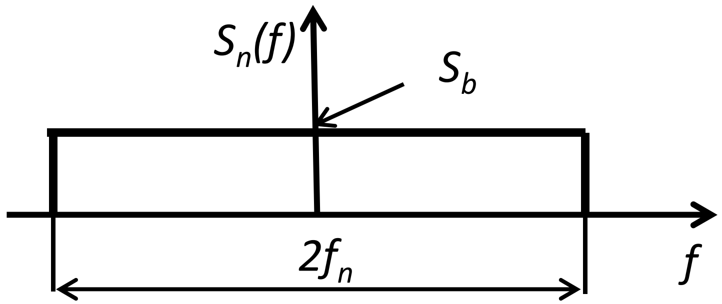
a) equal msq noise : $R_{nn}(0) = \overline{\boldsymbol{n^2}}$

b) equal spectral density: [area of $R_{nn}(\tau)$] = S_b ,

i.e. $\overline{n^2}\, 2\, T_n = S_b$

Correlation width = area/peak

$$\Delta\tau = 2T_n$$



in frequency: $S_n(f)$ rectang approx, half-width f_n

a) equal msq noise : [area of $S_n(f)$] = $\overline{\boldsymbol{n^2}}$

i.e. $S_b 2\, f_n = \overline{\boldsymbol{n^2}}$

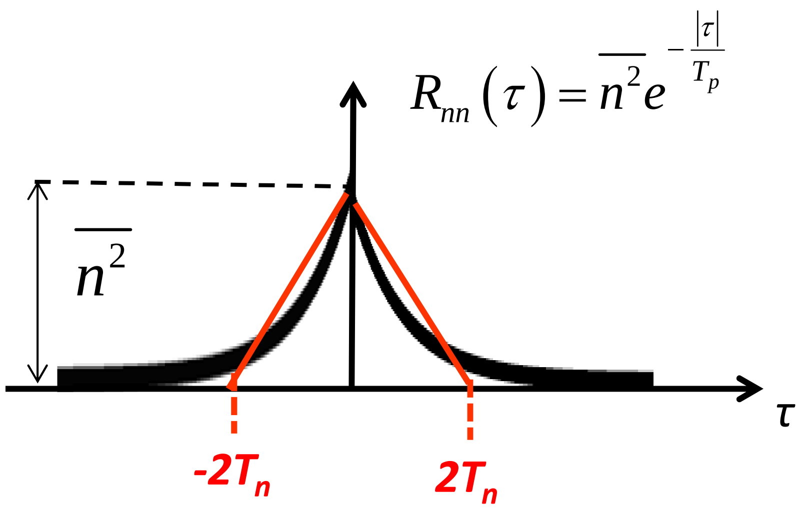
b) equal spectral density: $S_n(0) = S_b$

Noise bandwidth: area/peak

$$\Delta f = 2f_n$$

Note that $\Delta\tau \cdot \Delta f = 1$ which is consistent with $S_n(f) = F[R_{nn}(\tau)]$

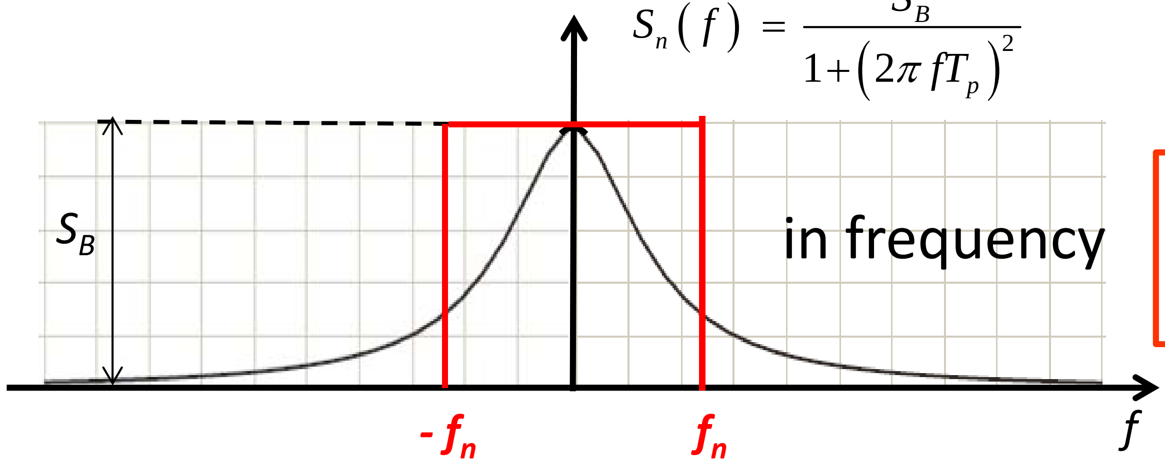
Simplified description of Lorentzian spectrum



$$S_B = \int_{-\infty}^{\infty} R_{nn}(\tau)\, d\tau = \overline{n^2}\, 2T_p$$

in time

$$T_n = T_p$$



$$\overline{n^2} = \int_{-\infty}^{\infty} S_n(f)\, df = S_B \frac{1}{2T_p}$$

in frequency

$$2 f_n = \frac{1}{2T_p}$$

Note that $f_n \neq f_p$, namely $f_n = \frac{1}{4T_p} = \frac{\pi}{2} f_p$

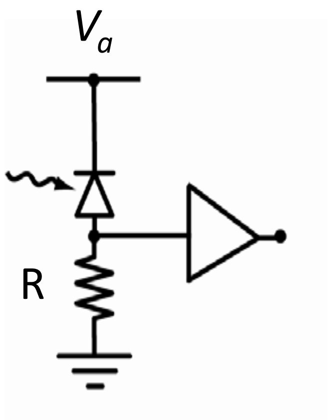
Foundations

of White-Noise Filtering

Noise filtering clarified by the Poisson pulse model

- Noise is a random superposition of elementary pulses
- The elementary pulse type (i.e. pulse waveform and its F-transform) defines the noise type (i.e. autocorrelation shape and spectrum shape)
- The passage through a linear constant-parameter filter modifies the elementary pulse type
- The pulse modification causes a corresponding modification of the noise
- Noise filtering can thus be understood, studied and evaluated by understanding, studying and evaluating the filtering of the elementary pulses

Low-pass filtering of White noise



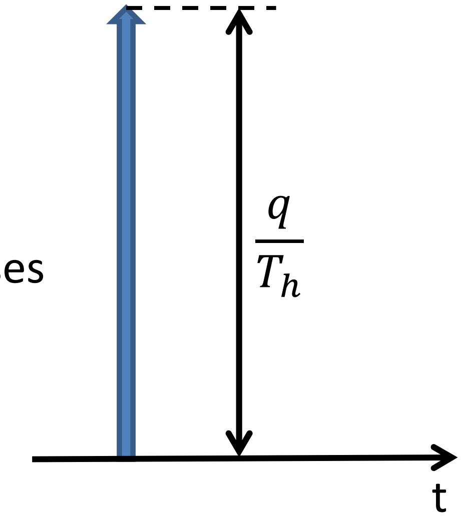
shot current white noise:

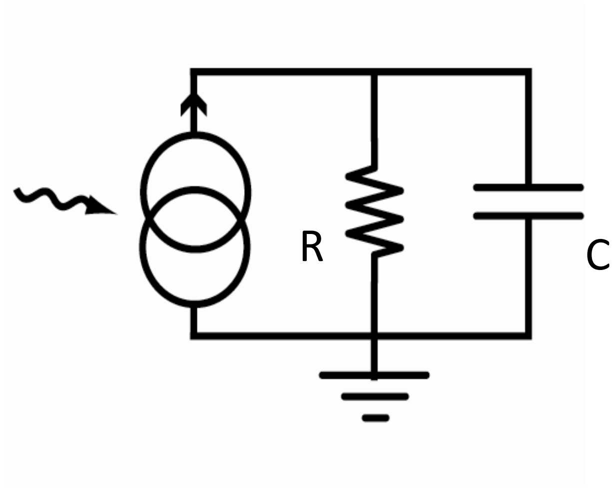
$S_i(f) \cong qI$ for *f<< 10GHz*

$R_{ii}(\tau) \cong qI\delta(\tau)$ for *τ >> 100ps*

diode current: elementary short pulses with rate $p = I/q$, $T_h \cong 100ps$ approximate δ-pulses

$qh(t) \cong q\delta(t)$



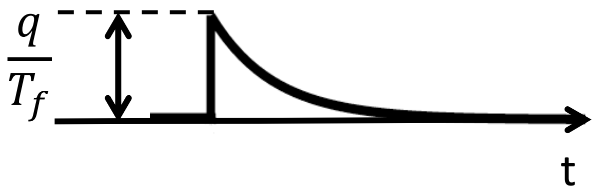


current in R: elementary exponential pulses with rate $p = I/q$, $T_f = RC$

$q \cdot f(t)$

$$f(t) = \frac{1}{T_f} 1(t) e^{-t/T_f}$$

$$F(f) = \frac{1}{1 + j2\pi f T_f}$$

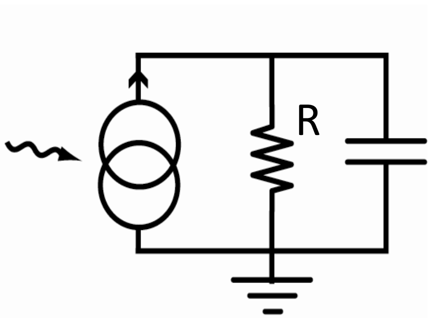


e.g. with *R= 100kΩ* and *C = 10pF* we have T_f *= 1µs*

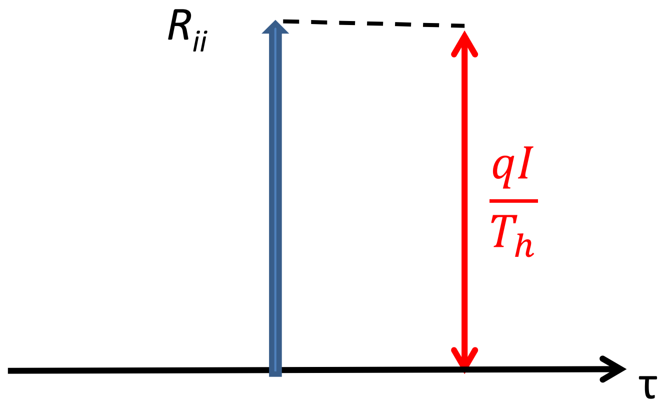
Low-pass filtering of White noise: time domain view

Input noise (current in the diode):
δ-like autocorrelation (width $\approx 100ps$)

$R_{ii}(\tau) \cong qI\delta(\tau)$ for *τ >> 100ps*



T_f = R C = 1μs

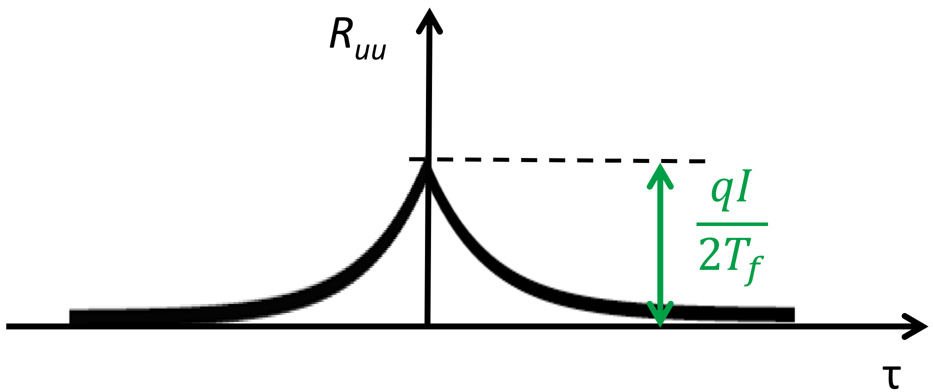


***To compare msq** values of noise **before** and **after** filtering*
*compare the **central values** of autocorrelation functions*

Output noise (current in R):
autocorrelation function

$$R_{uu}(\tau) = qI \cdot k_{ff}\,(\tau)$$

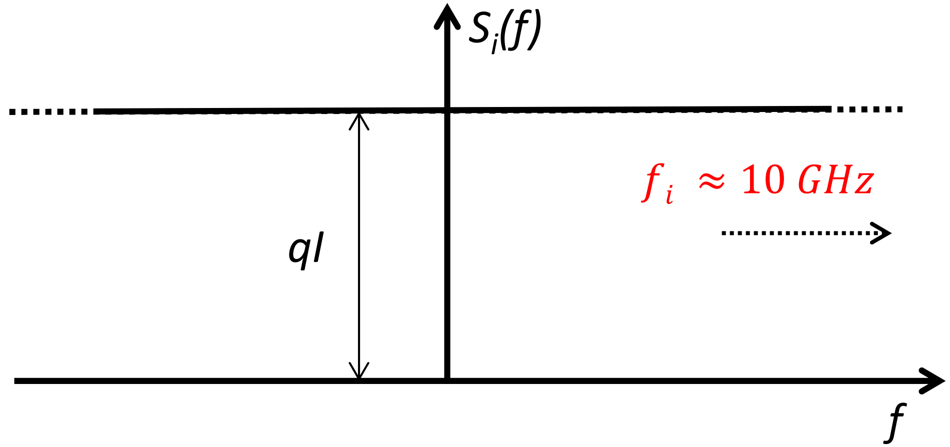
$$k_{ff}(\tau) = \frac{1}{2T_f}\, e^{-\left|\frac{\tau}{T_f}\right|}$$

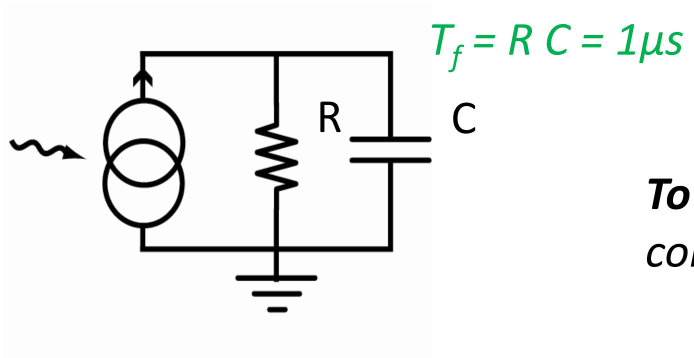


Low-pass filtering of White noise: frequency domain view

Input noise (diode current):
spectral density S_i constant
(bandwidth $f_i \approx 10\ GHz$)

$S_i(f) = S_b$ for f << 10 GHz



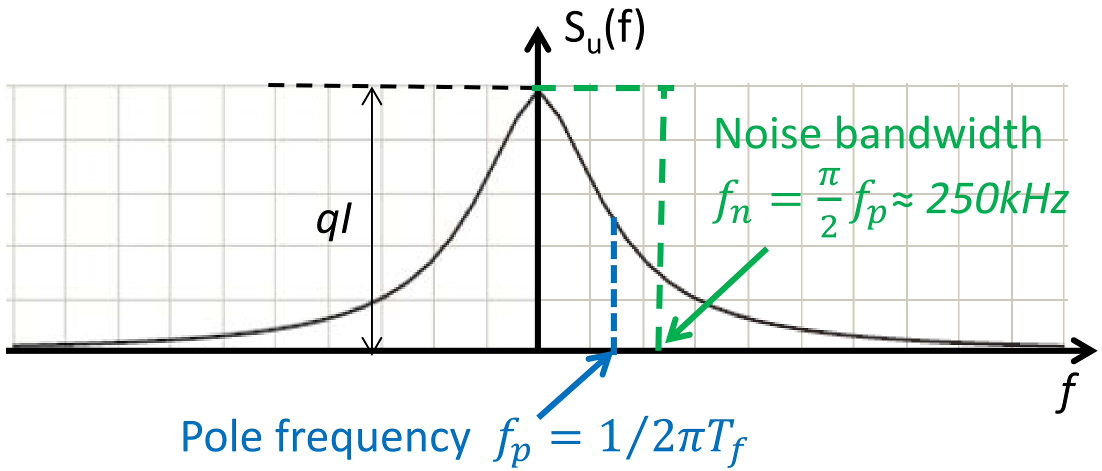


To compare *msq values of noise* ***before*** *and* ***after*** *filtering*
compare the ***areas*** *of input and output spectral densities*

Output noise (current in R):
spectral density function $S_u(f)$

$$S_u(f) = qI \cdot |F(f)|^2$$

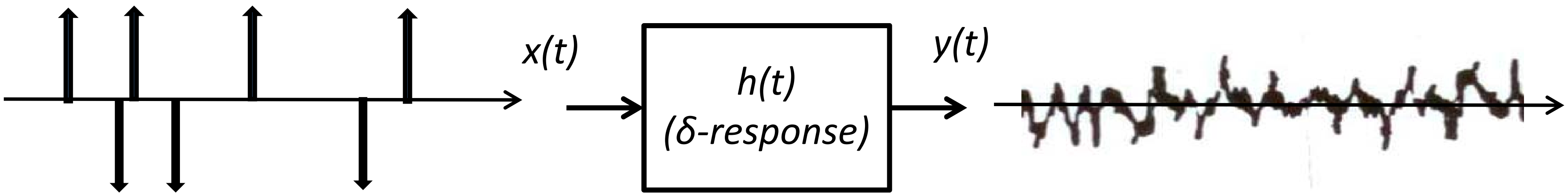
$$|F(f)|^2 = \frac{1}{1 + (2\pi f T_f)^2}$$



Generation and Simulation of Any Noise by a Poisson Process

Generation of any noise from Poisson process

Poisson process
(δ-pulses) → **Constant-Parameter Linear** filter → Output Noise



p pulse rate (p/s)
Q pulse area

Filter autocorrelation

$$k_{hh}(\tau) = \int_{-\infty}^{\infty} h(t)h(t+\tau)dt$$

Tranfer Function
H(f) = F[h(t)]

White Input Noise

$$\begin{cases} R_{xx}(\tau) = pQ^2\delta(\tau) \\ S_x(f) = pQ^2 \end{cases}$$

→

Output Noise

$$\begin{cases} R_{yy}(\tau) = pQ^2\, k_{hh}(\tau) \\ S_y(f) = pQ^2 \left|H(f)\right|^2 \end{cases}$$

Generation of any noise from Poisson process

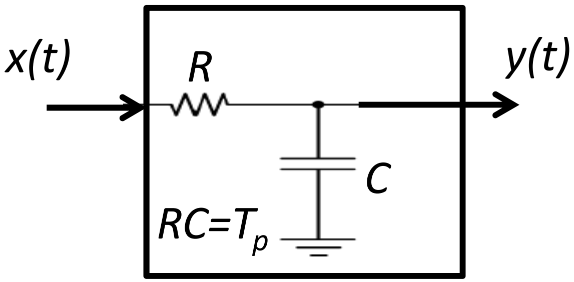
For producing a given $S_y(f)$ (i.e. a given $R_{yy}(\tau) = F^{-1}[S_y(f)]$) the filter must have

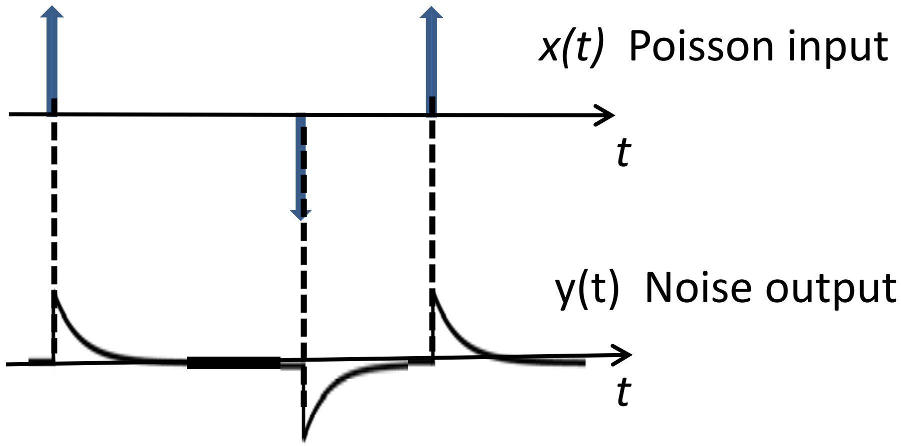
- $|H(f)|^2 = S_y(f)$ normalized to 1 at f=0
- $k_{hh}(\tau) = R_{yy}(\tau)$ normalized to unit area

Example: band-limited white noise $S_y = S_B \dfrac{1}{1+\left(f/f_p\right)^2}$ *with* $f_p = \dfrac{1}{2\pi T_p}$

$$H(\omega) = \frac{1}{1 + j\omega T_p}$$

$$h(t) = \frac{1}{T_p} e^{-\frac{t}{T_p}} 1(t)$$





$$S_y(f) = pQ^2 \left|H(f)\right|^2 = pQ^2 \frac{1}{1+\left(f/f_p\right)^2}$$

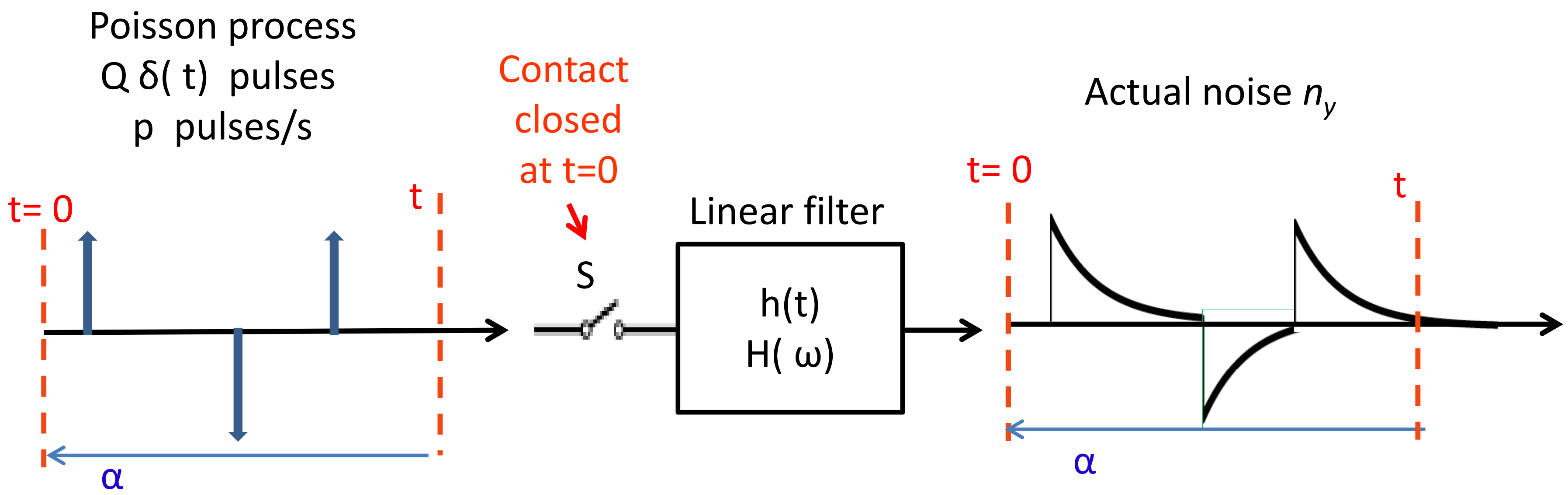
Appendix:
Noise Power Transients

Q: how does the noise power rise
when a noise source is switched on ?

A: the Poisson model clarifies!

Rise-time of the Noise power

Noise modeling by Poisson process also shows how noise rises after switch-on



Noise switch-on at t=0 modeled by closing at t=0 the switch at the filter input.

The mean square output at time t is computed by Campbell's theorem

but integrating **only over the interval where output pulses occur**, i.e. for **$0 < \alpha < t$**,

$$\overline{n_y^2(t)} = pQ^2 \int_0^t h^2(\alpha)\,d\alpha$$

Rise-time of real white noise

The rise of the output noise intensity can be observed by computing the root-mean square noise (rms) versus time *t* (normalized to unit spectral density of the input Poisson noise)

$$\sigma_y(t) = \sqrt{\frac{\overline{n_y^2(t)}}{pQ^2}} = \sqrt{\int_0^t h^2(\alpha)\,d\alpha}$$

Let's consider real White Noise with band-limit due to a simple pole with time constant T_p

- The noise is modeled by pulses $h(t) = \frac{1}{T_p} e^{-\frac{t}{T_p}} 1(t)$
- The rise of the rms noise is

$$\sigma_y(t) = \sqrt{\int_0^t \frac{1}{T_p^2} e^{-\frac{2\alpha}{T_p}} d\alpha} = \sqrt{\frac{1}{2T_p} \cdot \left(1 - e^{-\frac{2t}{T_p}}\right)}$$

Rise-time of real white noise

$$\sigma_y(t) = \sqrt{\int_0^t \frac{1}{T_p^2} e^{-\frac{2\alpha}{T_p}} d\alpha} = \sqrt{\frac{1}{2T_p} \cdot \left(1 - e^{-\frac{2t}{T_p}}\right)}$$

Wide-band «white» Noise

- The time constant T_p is short and the intensity (rms) rises swiftly, reaching in a few T_p the steady value

$$\sigma_y = \sqrt{\frac{1}{2T_p}} \qquad \text{for} \quad t \gg T_p$$

Moderately Wide-band Noise

- the time constant T_p is longer and the intensity rise is slower.
  In the first part $t \ll T_p$ and the intensity rises approximately as $\sqrt{t}$

$$\sigma_y(t) = \sqrt{\frac{1}{2T_p} \cdot \left(1 - e^{-\frac{2t}{T_p}}\right)} \approx \frac{\sqrt{t}}{T_p^2}$$

Rise-time of random-walk noise

«Random-Walk» Noise»

- denotes noise with spectral density $\propto 1/\omega^2$.
- is generated by integration of white noise
  (e.g. shot current noise $S_i = 2qI$ in capacitor C → voltage noise $S_v = 2qI/ C^2 \omega^2$)
- is modeled by step elementary pulses $h(t) = 1(t)$
- The variance rises as $\sqrt{t}$

$$\sigma_y(t) = \sqrt{\int_0^t h^2(\alpha)d\alpha} = \sqrt{\int_0^t d\alpha} = \sqrt{t}$$

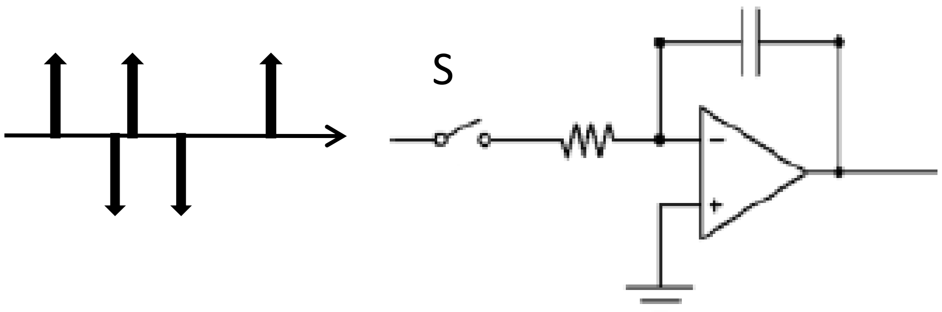
- At long time t → ∞ the variance is divergent
  In the frequency domain this corresponds to the power over a band extended down to very low frequency $f \to 0$

$$\sigma_y^2 = \lim_{f_i \to 0} \int_{f_i}^{\infty} |H(f)|^2 df = \lim_{f_i \to 0} \int_{f_i}^{\infty} \frac{1}{f^2} df \to \infty$$

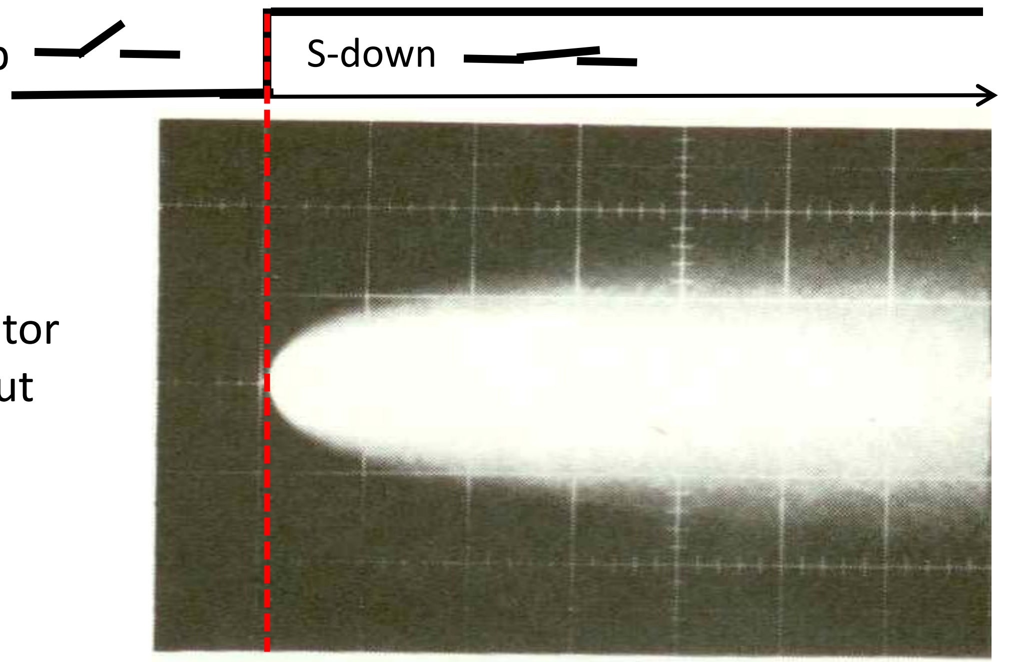
Rise-time of random-walk noise

Poisson process
(δ-pulses)

Integrator
h(t)=1(t)



Random-walk noise
y(t)



Integrator
output
y(t)

$$\sigma_y(t) = \sqrt{t}$$