Sensors, Signals and Noise

COURSE OUTLINE

- Introduction
- Signals and Noise
- Filtering Noise
- Sensors and associated electronics

Processing Noise with Linear Filters

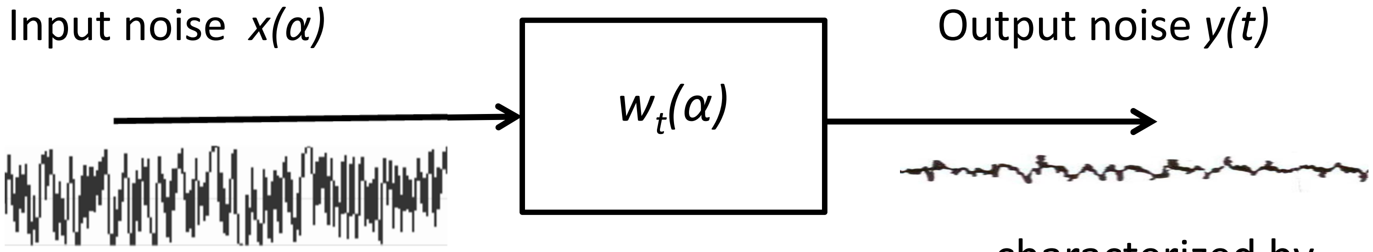
- Mathematical Foundations
- Filtering Stationary Noise
- Filtering White Noise
- Filtering Noise with Constant-Parameter Filters

and for those who want to gain a better insight

- Appendix: Input-Output Crosscorrelation and Autocorrelation
  with Stationary Noise and Constant-Parameter Filters

Mathematical Foundations of Noise Processing by Linear Filters

Noise filtering



characterized by

$$R_{xx}(\alpha, \alpha+\gamma) = \overline{x(\alpha)x(\alpha+\gamma)}$$

characterized by

$$R_{yy}(t, t+\tau) = \overline{y(t)y(t+\tau)}$$

The **output** autocorrelation can be obtained in terms of the **input** autocorrelation and of the filter **weighting** function :

$$R_{yy}(t_1, t_2) = \overline{y(t_1)y(t_2)} =$$

$$= \overline{\int_{-\infty}^{\infty} x(\alpha)w_1(\alpha)d\alpha \cdot \int_{-\infty}^{\infty} x(\beta))w_2(\beta)d\beta} =$$

$$= \iint_{-\infty}^{\infty} \overline{x(\alpha)x(\beta)} \cdot w_1(\alpha)w_2(\beta)d\alpha\, d\beta =$$

$$= \iint_{-\infty}^{\infty} R_{xx}(\alpha, \beta)\, w_1(\alpha)w_2(\beta)d\alpha\, d\beta$$

Noise filtering

The **output autocorrelation**

$$R_{yy}(t_1, t_2) = R_{yy}(t_1, t_1 + \tau) = \overline{y(t_1)\, y(t_1 + \tau)}$$

by setting in evidence the intervals of autocorrelation at the input $\gamma = \beta - \alpha$ and at the output $\tau = t_2 - t_1$ can be expressed as

$$\boxed{R_{yy}(t_1, t_1 + \tau) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \alpha + \gamma)\, w_1(\alpha) w_2(\alpha + \gamma) d\alpha\, d\gamma}$$

and in particular the **mean square noise** at time t_1 is

$$\boxed{\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \alpha + \gamma)\, w_1(\alpha)\, w_1(\alpha + \gamma) d\alpha\, d\gamma}$$

NB: these equations are **valid for all cases of noise and linear filtering**, that is, also for non-stationary input noise and for time-variant filters.

Filtering Stationary Noise

Filtering Stationary Noise

In case of stationary noise the input autocorrelation depends only on the time interval γ

$$R_{xx}(\alpha, \alpha + \gamma) = R_{xx}(\gamma)$$

The output autocorrelation is correspondingly simplified

$$R_{yy}(t_1, t_1 + \tau) = \iint_{-\infty}^{\infty} R_{xx}(\gamma)\, w_1(\alpha) w_2(\alpha + \gamma) d\alpha\, d\gamma =$$

$$= \int_{-\infty}^{\infty} R_{xx}(\gamma)\, d\gamma \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha + \gamma) d\alpha$$

NB: with stationary input noise:

a) a constant parameter filter produces stationary output noise.

b) a time-variant filter can produce a non-stationary output noise!

Filtering Stationary Noise

$$R_{yy}(t_1, t_1+\tau) = \int_{-\infty}^{\infty} R_{xx}(\gamma)\, d\gamma \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha+\gamma) d\alpha$$

Denoting by $k_{12w}(\gamma)$ the **cross**correlation of the weighting functions $w_1(\alpha)$ and $w_2(\alpha)$

$$k_{12w}(\gamma) = \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha+\gamma) d\alpha$$

We can write

$$\boxed{R_{yy}(t_1, t_1+\tau) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{12w}(\gamma)\, d\gamma}$$

For the **mean square noise** we must consider the **auto**correlation $k_{11w}(\alpha)$ of $w_1(\alpha)$

$$\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \int_{-\infty}^{\infty} R_{xx}(\gamma)\, d\gamma \int_{-\infty}^{\infty} w_1(\alpha) w_1(\alpha+\gamma) d\alpha$$

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma)\, d\gamma}$$

Filtering Stationary Noise

With **stationary input noise** and for **any linear filter** (i.e. both constant-parameter and time variant filters) the **output noise** mean square value can be computed

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma)\, d\gamma}$$

By the Parseval theorem extension and recalling that

$$F[k_{11w}(\gamma)] = |W_1(f)|^2$$

the output mean square noise can be computed also in the frequency domain

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_1(f)|^2\, df}$$

Filtering Stationary Noise

The mean square output of a filter that receives stationary noise can be computed

in the time domain as

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma)\, d\gamma}$$

in the frequency domain as

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_1(f)|^2\, df}$$

and in case of **white** noise, i.e. with

$$R_{xx}(\gamma) = S_b \cdot \delta(\gamma) \qquad\qquad S_x(f) = S_b$$

it is simply

$$\boxed{\overline{y^2(t_1)} = S_b \cdot k_{11w}\,(0) = S_b \int_{-\infty}^{\infty} {w_1}^2(\alpha) d\alpha}$$

$$\boxed{\overline{y^2(t_1)} = S_b \int_{-\infty}^{\infty} |W_1(f)|^2\, df}$$

Filtering White Noise

Filtering White NON-Stationary noise

The fact that a White NON-Stationary noise has δ-like autocorrelation

$$R_{xx}(\alpha, \alpha+\gamma) = S_b(\alpha)\delta(\gamma)$$

brings simplification to the equation of the output autocorrelation

$$R_{yy}(t_1, t_1+\tau) = \iint_{-\infty}^{\infty} S_b(\alpha)\delta(\gamma)\cdot w_1(\alpha) w_2(\alpha+\gamma) d\alpha\, d\gamma$$

$$\boxed{R_{yy}(t_1, t_1+\tau) = \int_{-\infty}^{\infty} S_b(\alpha)\cdot w_1(\alpha) w_2(\alpha) d\alpha}$$

and to the equation of the output mean square value

$$\boxed{\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \int_{-\infty}^{\infty} S_b(\alpha)\cdot {w_1}^2(\alpha) d\alpha}$$

the equation is conceptually similar to that for $\overline{y^2}$ *in discrete-time filtering, with samples x taken at clocked times* α_i *and multiplied by weights* w_i *and summed*

$$\overline{y^2(t_1)} = \sum_{i=0}^{n} \overline{x^2(\alpha_i)}\cdot {w_i}^2$$

Filtering White Stationary noise

The fact that White Stationary noise has constant intensity (power)

$$R_{xx}(\alpha, \alpha+\gamma) = R_{xx}(\gamma) = S_b \cdot \delta(\gamma)$$

further simplifies the equation of the output autocorrelation

$$\boxed{R_{yy}(t_1, t_1+\tau) = S_b \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha) d\alpha = S_b \cdot k_{12w}(0)}$$

and of the output mean square value

$$\boxed{\overline{y^2(t_1)} = S_b \cdot k_{11w}(0) = S_b \int_{-\infty}^{\infty} {w_1}^2(\alpha) d\alpha}$$

the equation is similar to that for discrete-time filtering of stationary white input noise

$$\overline{y^2(t_1)} = \overline{x^2} \sum_{i=0}^{n} {w_i}^2$$

By Parseval theorem we have also

$$\boxed{\overline{y^2(t_1)} = S_b \int_{-\infty}^{\infty} |W_1(f)|^2 df}$$

Filtering Noise with Constant-Parameter Filters

About CONSTANT-PARAMETER filters

The constant-parameter filters:

- are completely characterized by the δ-response *h(t)* in time and by the transfer function *H(f) = F[h(t)]* in the frequency domain
- have weighting $w_m(\alpha)$ for acquisition at time t_m simply related to the δ-response

$$w_m(\alpha) = h(t-\alpha)$$

- therefore have

$$|W_m(f)|^2 = |H(f)|^2$$

- They are PERMUTABLE. In a cascade of constant parameter filters, if the order of the various filters in the sequence is changed, the final output does NOT change.

✓ *NB1: this is absolutely true in principle, but it is limited in practical implementation by the limitations of linear behaviour (e.g. the finite dynamical range of real circuits).*

- They are REVERSIBLE. A constant parameter filter can change the shape of a signal, but it is always possible to find a restoring filter, that is, another constant parameter filter which restores the signal to the original shape.

✓ *NB2: this is absolutely true in principle, but it is limited in practical implementation by the limitations of linear behaviour and by the presence of noise (i.e. higher noise can be associated to the restored signal)*

CONSTANT-PARAMETER filters with NON-stationary input noise

The output autocorrelation is

$$R_{yy}(t_1, t_2) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \beta)\, w_1(\alpha) w_2(\beta) d\alpha\, d\beta =$$

$$= \iint_{-\infty}^{\infty} R_{xx}(\alpha, \beta) \cdot h(t_1 - \alpha) h(t_2 - \beta) d\alpha\, d\beta =$$

$$= \int_{-\infty}^{\infty} h(t_1 - \alpha) d\alpha \int_{-\infty}^{\infty} R_{xx}(\alpha, \beta) \cdot h(t_2 - \beta) d\beta =$$

That is

$$\boxed{R_{yy}(t_1, t_2) = R_{xx}(\alpha, \beta) * h(\beta) * h(\alpha)}$$

CONSTANT-PARAMETER filters with Stationary input noise

Starting from

$$R_{yy}(t_1, t_2) = R_{yy}(t_1 + \tau) = R_{xx}(\alpha, \beta) * h(\beta) * h(\alpha)$$

and taking into account that:

- the stationary input autocorrelation depends only on the interval $\gamma = \beta - \alpha$
- $d\beta = d\gamma$
- $d\alpha = -\, d\gamma$
- the output autocorrelation is also stationary and depends only on the interval τ

we can obtain (detailed equations available in Appendix)

$$\boxed{R_{yy}(\tau) = R_{xx}(\gamma) * h(\gamma) * h(-\gamma) = R_{xx}(\gamma) * k_{hh}(\gamma)}$$

and therefore

$$\boxed{S_y(f) = S_x(f) \cdot |H(f)|^2}$$

CONSTANT-PARAMETER filters with Stationary input noise

From the output autocorrelation $R_{yy}(\tau) = R_{xx}(\gamma) * k_{hh}(\gamma)$ we obtain for the output mean square value

$$\boxed{\overline{y^2} = R_{yy}(0) = \int_{-\infty}^{\infty} R_{xx}(\gamma)\, k_{hh}(\gamma) d\gamma}$$

and by Parseval's theorem

$$\boxed{\overline{y^2} = \int_{-\infty}^{\infty} S_x(f) |H(f)|^2 df}$$

In the case of **white** input noise $R_{xx}(\gamma) = S_b \delta(\gamma)$ and therefore

$$\boxed{\overline{y^2} = S_b\, k_{hh}(0)}$$

$$\boxed{\overline{y^2} = S_b \int_{-\infty}^{\infty} |H(f)|^2 df}$$

Appendix:
output-input cross-correlation and output autocorrelation with constant parameter filters and stationary noise

Appendix: output-input cross-correlation with constant parameter filters and stationary noise

Let us see first the output-input crosscorrelation $R_{yx}(\tau)$

$$R_{yx}(t_1,t_2) = \overline{y(t_1)\,x(t_2)} = \overline{\int_{-\infty}^{\infty} x(\alpha)\,h(t_1-\alpha)\,x(t_2)\,d\alpha} = \int_{-\infty}^{\infty} \overline{x(\alpha)\,x(t_2)}\,h(t_1-\alpha)\,d\alpha =$$

$$= \int_{-\infty}^{\infty} R_{xx}(\alpha,t_2)\,h(t_1-\alpha)\,d\alpha = \int_{-\infty}^{\infty} R_{xx}(t_2-\alpha)\,h(t_1-\alpha)\,d\alpha$$

and setting

$$t_2-\alpha=\gamma \qquad d\alpha=-d\gamma \qquad \tau=t_2-t_1$$

$$R_{yx}(t_1,t_2) = R_{yx}(t_1,t_1+\tau) = -\int_{\infty}^{-\infty} R_{xx}(\gamma)\,h(t_1-t_2+\gamma)\,d\gamma =$$

$$= \int_{-\infty}^{\infty} R_{xx}(\gamma)\,h(-\tau+\gamma)\,d\gamma = \int_{-\infty}^{\infty} R_{xx}(\gamma)\,h\big(-(\tau-\gamma)\big)\,d\gamma$$

we see that

$$\boxed{R_{yx}(\tau) = R_{xx}(\tau) * h(-\tau)}$$

Appendix: input-output cross-correlation with constant parameter filters and stationary noise

Let us see now the input-output crosscorrelation $R_{xy}(\tau)$

$$R_{xy}(t_1,t_2)=\overline{x(t_1)\,y(t_2)}=\overline{\int_{-\infty}^{\infty}x(t_1)\,x(\beta)\,h(t_2-\beta)\,d\beta}=\int_{-\infty}^{\infty}\overline{x(t_1)\,x(\beta)}\,h(t_2-\beta)\,d\beta=$$

$$=\int_{-\infty}^{\infty}R_{xx}(t_1,\beta)\,h(t_2-\beta)\,d\beta=\int_{-\infty}^{\infty}R_{xx}(\beta-t_1)\,h(t_2-\beta)\,d\beta$$

and setting $\gamma=\beta-t_1\,;\quad d\gamma=d\beta\quad;\quad \tau=t_2-t_1$

$$R_{xy}(t_1,t_2)=R_{xy}(t_1,t_1+\tau)=\int_{-\infty}^{\infty}R_{xx}(\gamma)\,h(t_2-t_1-\gamma)\,d\gamma=$$

$$=\int_{-\infty}^{\infty}R_{xx}(\gamma)\,h(\tau-\gamma)\,d\gamma$$

We see that

$$\boxed{R_{xy}(\tau)=R_{xx}(\tau)*h(\tau)}$$

Appendix: output auto-correlation with constant parameter filters and stationary noise

$$R_{yy}(t_1,t_2)=\overline{y(t_1)\,y(t_2)}=\overline{y(t_1)\int_{-\infty}^{\infty}x(\beta)\,h(t_2-\beta)\,d\beta}=\int_{-\infty}^{\infty}\overline{y(t_1)\,x(\beta)}\,h(t_2-\beta)\,d\beta=$$

$$=\int_{-\infty}^{\infty}R_{yx}(t_1,\beta)\,h(t_2-\beta)\,d\beta=\int_{-\infty}^{\infty}R_{yx}(\beta-t_1)\,h(t_2-\beta)\,d\beta$$

and setting $\gamma=\beta-t_1\,;\quad d\gamma=d\beta\quad;\quad \tau=t_2-t_1$

$$R_{yy}(t_1,t_2)=R_{yy}(t_1,t_1+\tau)=\int_{-\infty}^{\infty}R_{yx}(\gamma)\,h(t_2-t_1-\gamma)\,d\gamma=$$

$$=\int_{-\infty}^{\infty}R_{yx}(\gamma)\,h(\tau-\gamma)\,d\gamma=R_{yx}(\tau)*h(\tau)=\left[R_{xx}(\tau)*h(-\tau)\right]*h(\tau)=$$

$$=R_{xx}(\tau)*\left[h(-\tau)*h(\tau)\right]$$

and finally

$$\boxed{R_{yy}(\tau)=R_{xx}(\tau)*k_{hh}(\tau)}$$