Sensors, Signals and Noise

COURSE OUTLINE

- Introduction
- Signals and Noise
- Filtering: Band-Pass Filters 3 – BPF3
- Sensors and associated electronics

Band-Pass Filtering 3

- Asynchronous Measurement of Sinusoidal Signals
- Principle of Synchronous Measurements of Sinusoidal Signals
- Noise Filtering in Synchronous Measurements
- Lock-in Amplifier Principle and Weighting Function
- Stage-by-stage view of Signals in the Lock-in Amplifier
- Stage-by-stage view of Noise in the Lock-in Amplifier
- Bandwidth, Response Time and S/N of the Lock-in Amplifier

Asynchronous Measurement of Sinusoidal Signals

Asynchronous measurement of sinusoidal signals

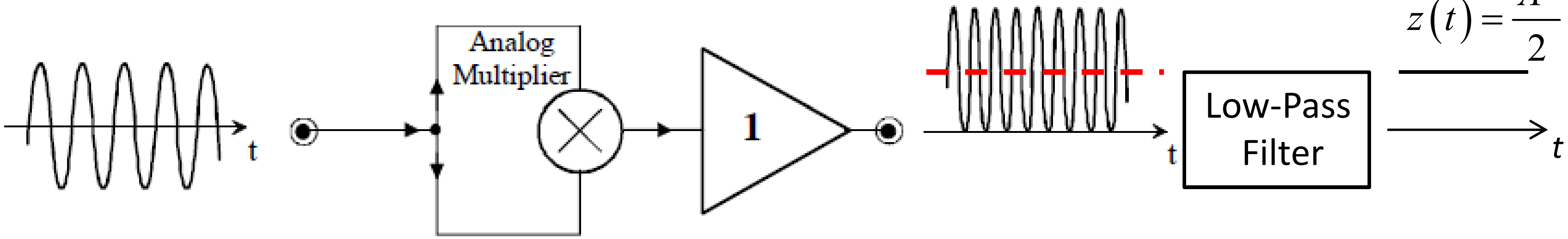
- Asynchronous (or phase-insensitive) techniques were devised for measuring a sinusoidal signal without needing an auxiliary **reference** that points out the peaking time (i.e. the phase of the signal).
- They are currently employed in AC voltmeters and amperometers.
- The basic circuits of such meters are
  - the mean-square detector
  - the half-wave rectifier
  - the full-wave rectifier
- For a correct measurement of the amplitude of the sinusoidal signal, it is necessary to avoid feeding a DC component to the input of an asynchronous meter circuit. Therefore, the meter must be preceded by a filter that cuts off the low-frequencies, that is, a band-pass or a high-pass filter.

Asynchronous measurement of sinusoidal signals with Mean-Square Detector

$$x(t) = A\cos(\omega t + \vartheta)$$

$$y(t) = A^2\cos^2(\omega t + \vartheta) = \frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega t + 2\vartheta)$$

$$z(t) = \frac{A^2}{2}$$

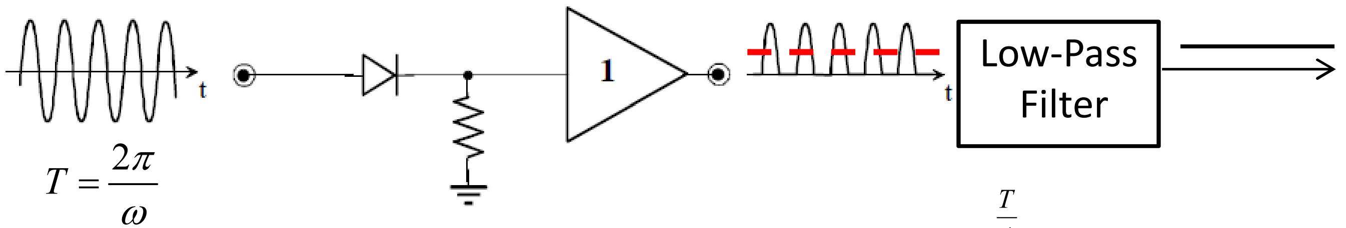


- It is a power-meter: the output is a measure of the **total** input mean power, sum of signal power (proportional to the square of amplitude A^2) plus noise power.
- The low-pass filter has NO EFFECT OF NOISE REDUCTION. In fact, it does not average the input, it averages **the square** of the input.
- For improving the S/N it is necessary to insert a filter **before** the Mean-Square Detector

Asynchronous measurement of sinusoidal signals with Rectifier

Half-Wave Rectifier (HWR)

$$x(t) = A\cos(\omega t)$$



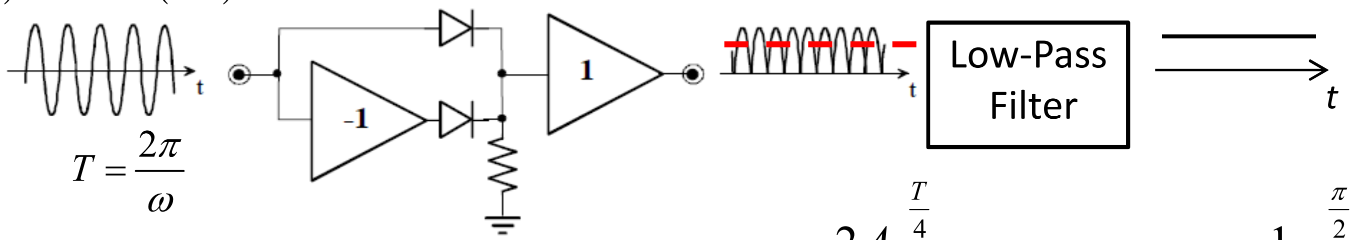
$$T = \frac{2\pi}{\omega}$$

$$z(t) = \frac{A}{\pi}$$

$$z = \frac{A}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos(\omega t)\,dt = A\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\varphi = \frac{A}{\pi}$$

Full-Wave Rectifier (FWR)

$$x(t) = A\cos(\omega t)$$



$$T = \frac{2\pi}{\omega}$$

$$z(t) = \frac{2A}{\pi}$$

$$z = \frac{2A}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos(\omega t)\,dt = 2A\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\varphi = \frac{2A}{\pi}$$

Asynchronous measurement of sinusoidal signals with Rectifier

- The measurement with a rectifier is not really asynchronous, it is **self-synchronized**. The sinusoidal signal itself decides when it has to be passed with positive polarity and when passed with negative polarity (in the full-wave rectifier) or not passed at all (in the half-wave rectifier).
- In such operation, the LPF reduces the contribution of the wide-band noise, thus improving the output S/N. However, this is true only if the input signal is remarkably higher than the noise, i.e. if the input S/N is high.
- As the input signal is reduced the noise gains increasing influence on the switching time of the rectifier, which progressively loses synchronism with the signal and tends to be synchronized with the zero-crossings of the noise.
- The loss of synchronization progressively degrades the noise reduction by the LPF. With moderate S/N the improvement due to LPF is modest; with low S/N it is very weak. With S/N < 1 there is no improvement, there is not even a measure of the signal: the output is a measure of the noise mean absolute value.
- In conclusion, meters based on rectifiers can just improve an already good S/N. They can't help to improve a modest S/N and it is out of the question to use them when S/N <1. For improving S/N it is necessary to employ filters **before** the meter.

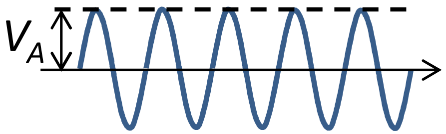
Synchronous (or Phase-Sensitive) Measurements of Sinusoidal Signals

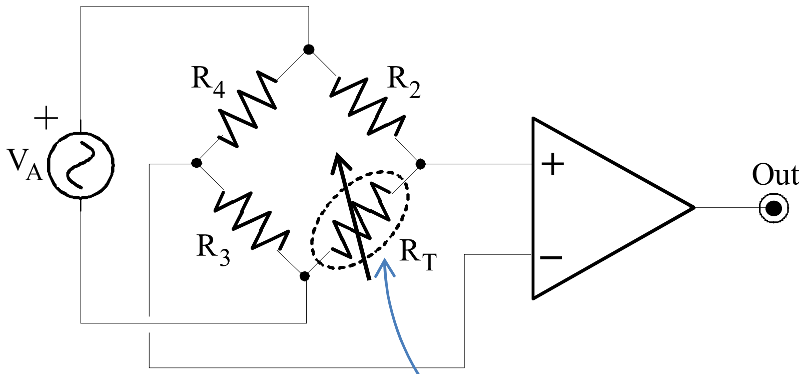
Signal and Reference for Synchronization

KEY EXAMPLE

for the study of synchronous measurements and narrow-band filtering

AC voltage supply





SIGNAL OUT

$$x(t) = A\cos(\omega_m t + \varphi)$$

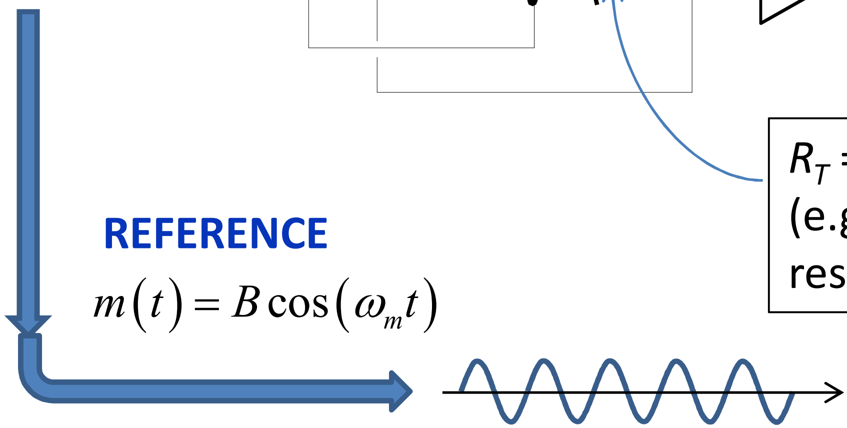
A to be measured

φ constant and known*

$R_T = R_T(\vartheta)$ resistance tracks a variable ϑ, (e.g. a temperature or a strain); resistances R_2 , R_3 , R_4 are constant

REFERENCE

$$m(t) = B\cos(\omega_m t)$$



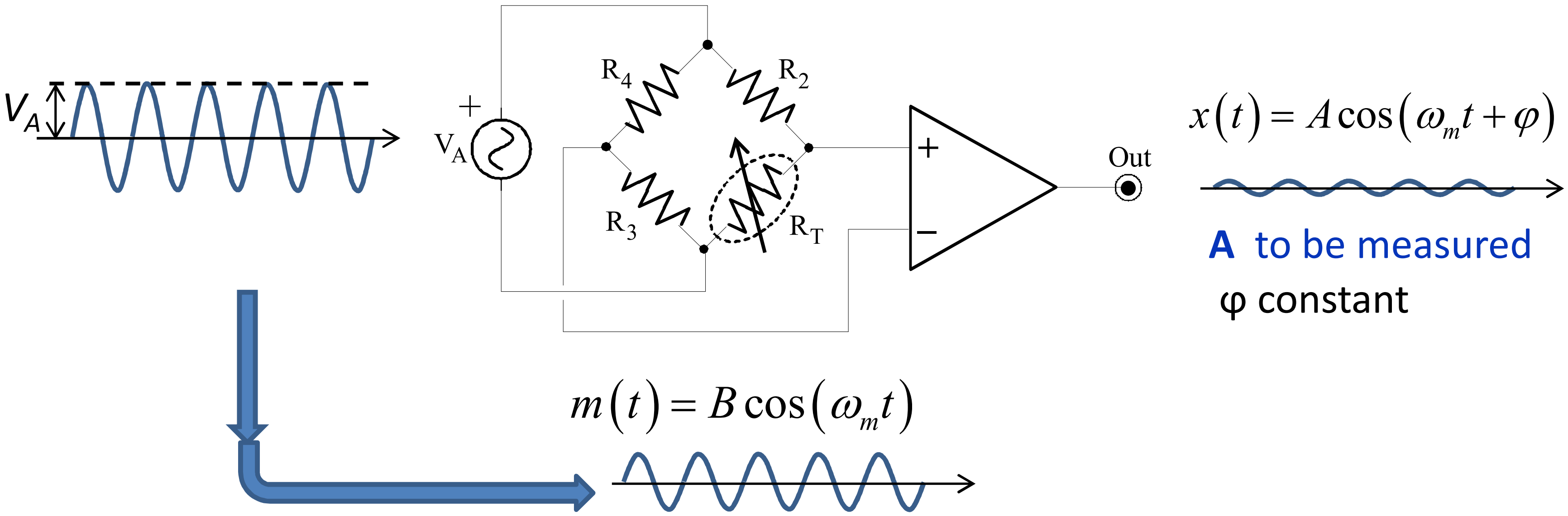
Shows the frequency and phase of the signal
i.e. points out the peak instants of the signal

* in this example φ=0 since the preamp passband limit is much higher than the signal frequency f_m

Signal and Reference for Synchronization

KEY EXAMPLE

for the study of synchronous measurements and narrow-band filtering



A to be measured

φ constant

R_T e.g. strain sensor, the resistance varies following a mechanical **strain** $\boldsymbol{\vartheta}$

a) in cases with **constant** strain ϑ

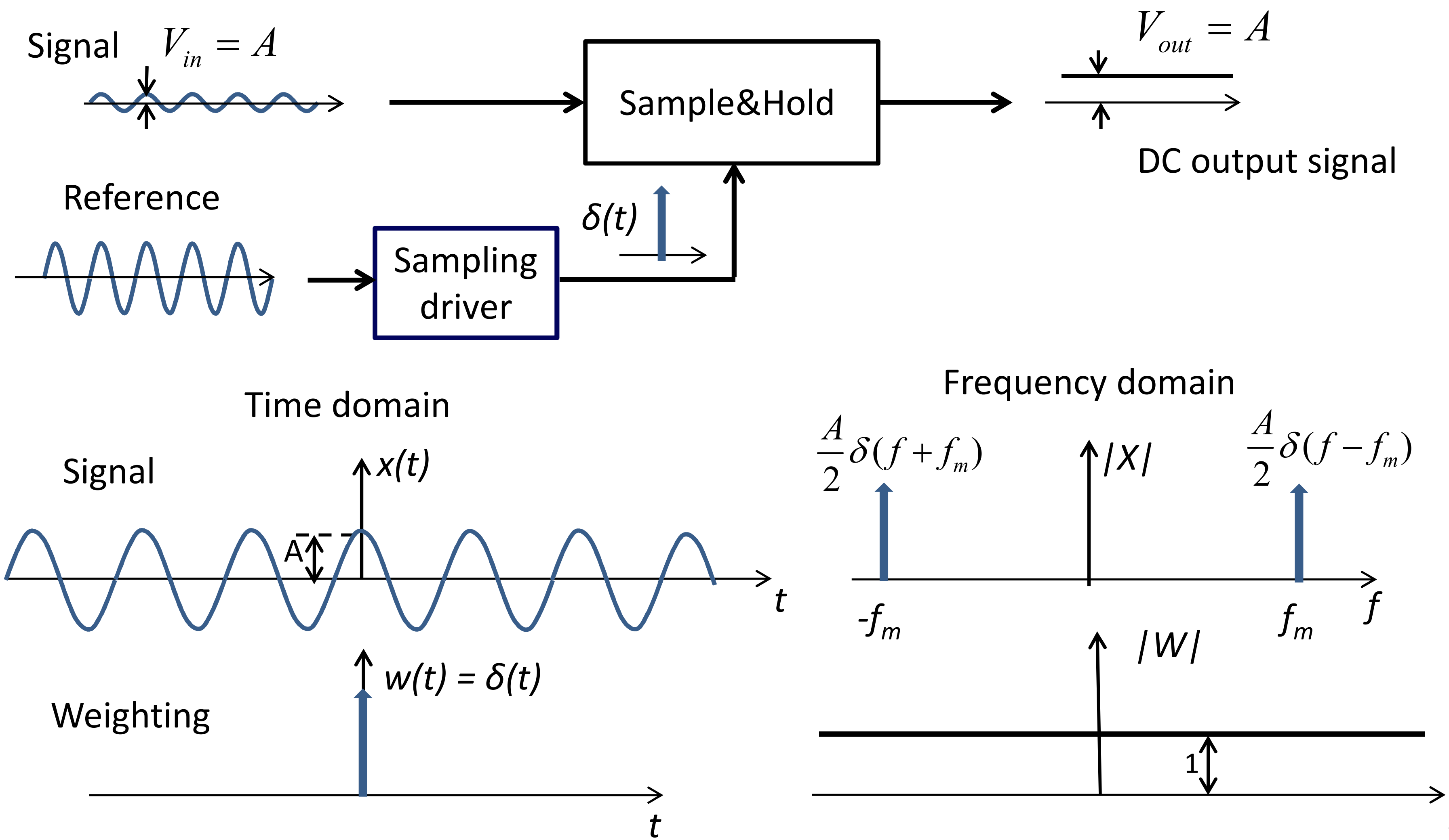
constant *A* → x(t) is a **pure sinusoidal** signal

b) in cases with **slowly variable** strain $\vartheta = \vartheta(t)$

variable $A = A(t)$ → x(t) is a **modulated sinusoidal** signal

SLOW variations = the Fourier components of $A(f)=F[A(t)]$ have frequencies $f \ll f_m$

Elementary synchronous measurement: peak sampling



NO FILTERING ACTION: output noise power = full input noise power

Noise Filtering
in Synchronous Measurements

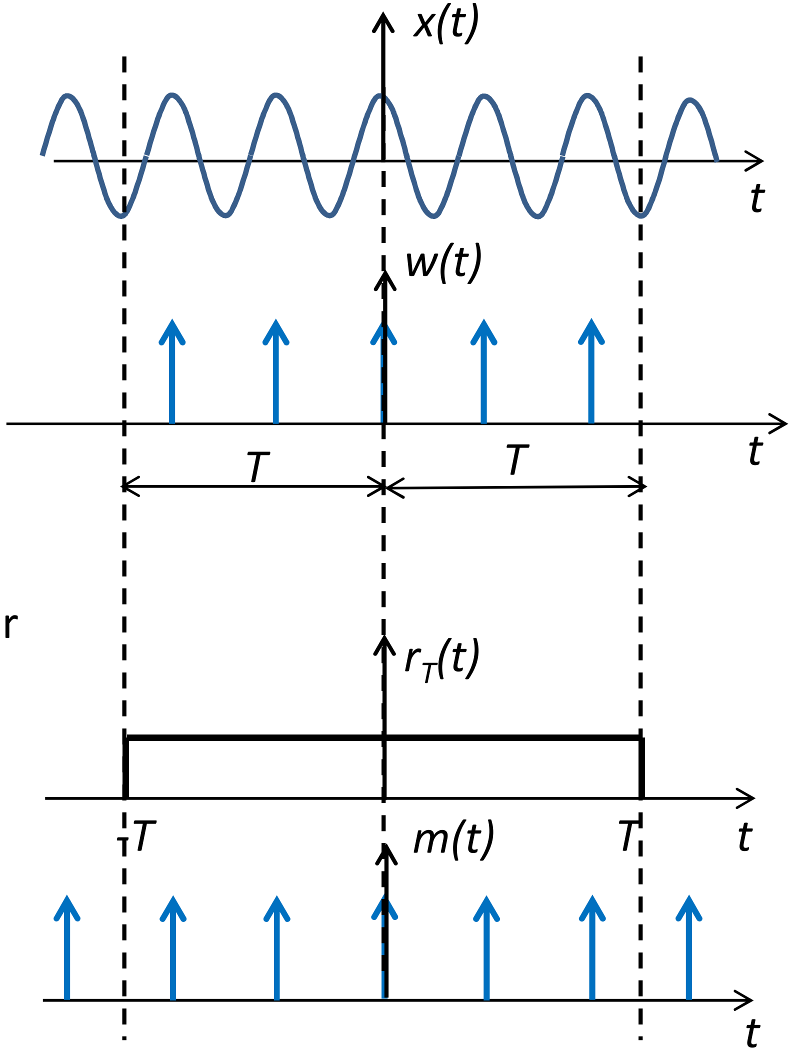
Synchronous measurement with averaging over many samples **N >>1** of the peak

$$x(t) = A\cos(2\pi f_m t)$$

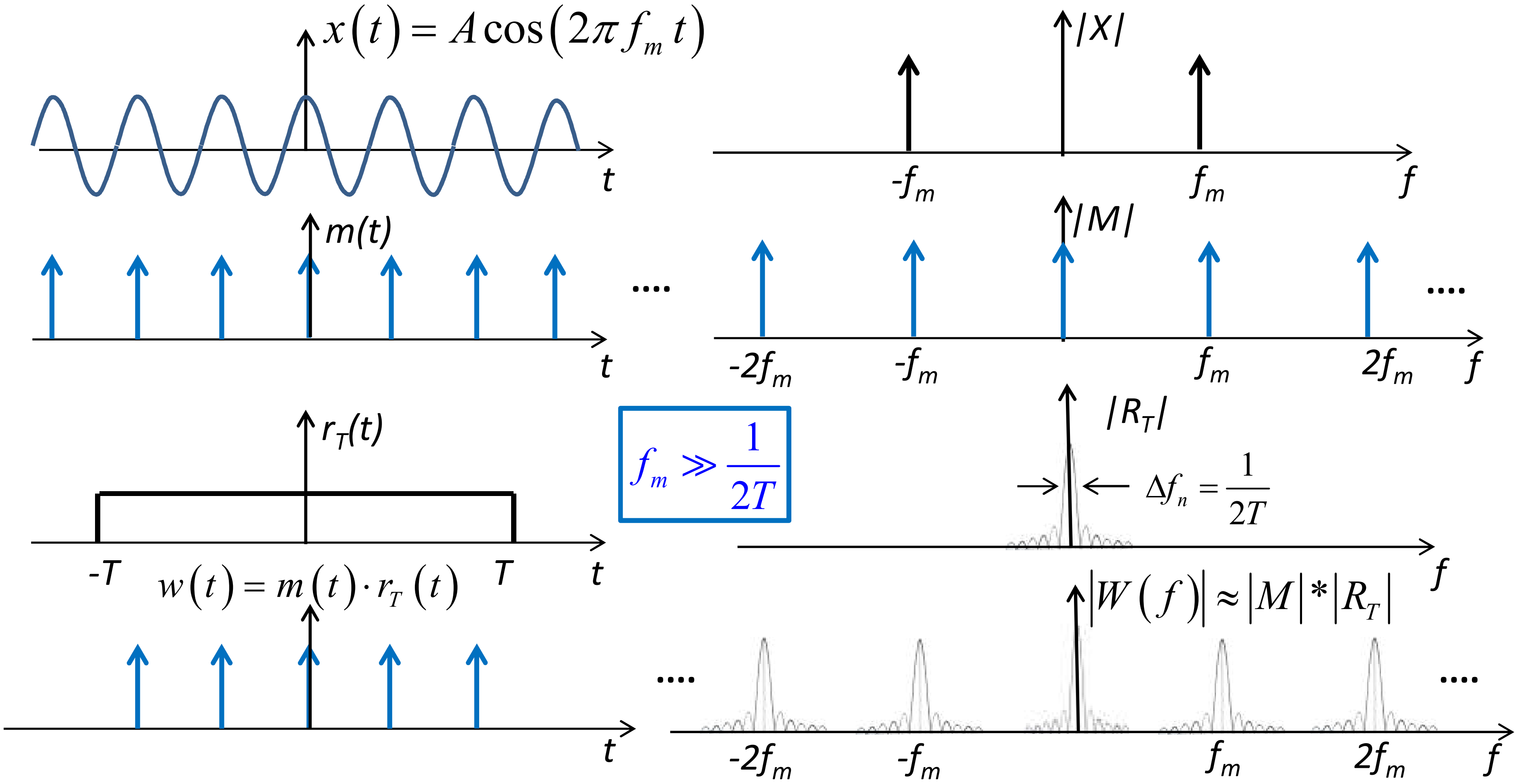
$$N = f_m 2T \gg 1$$

To take N samples
is equivalent to gate
a free-running sampler

$$w(t) = m(t) \cdot r_T(t)$$

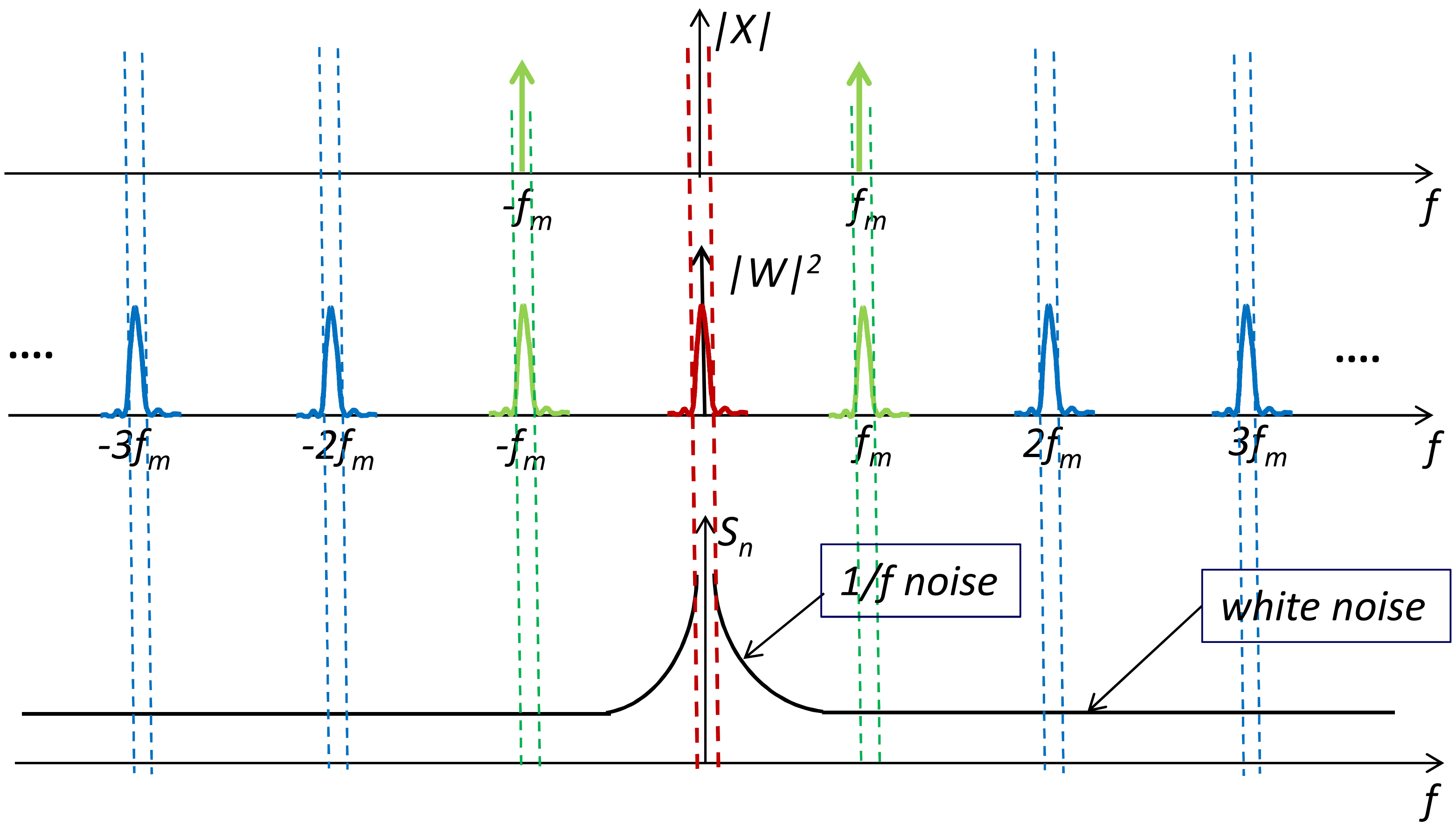


Synchronous measurement with averaging over many samples **N >>1** of the peak



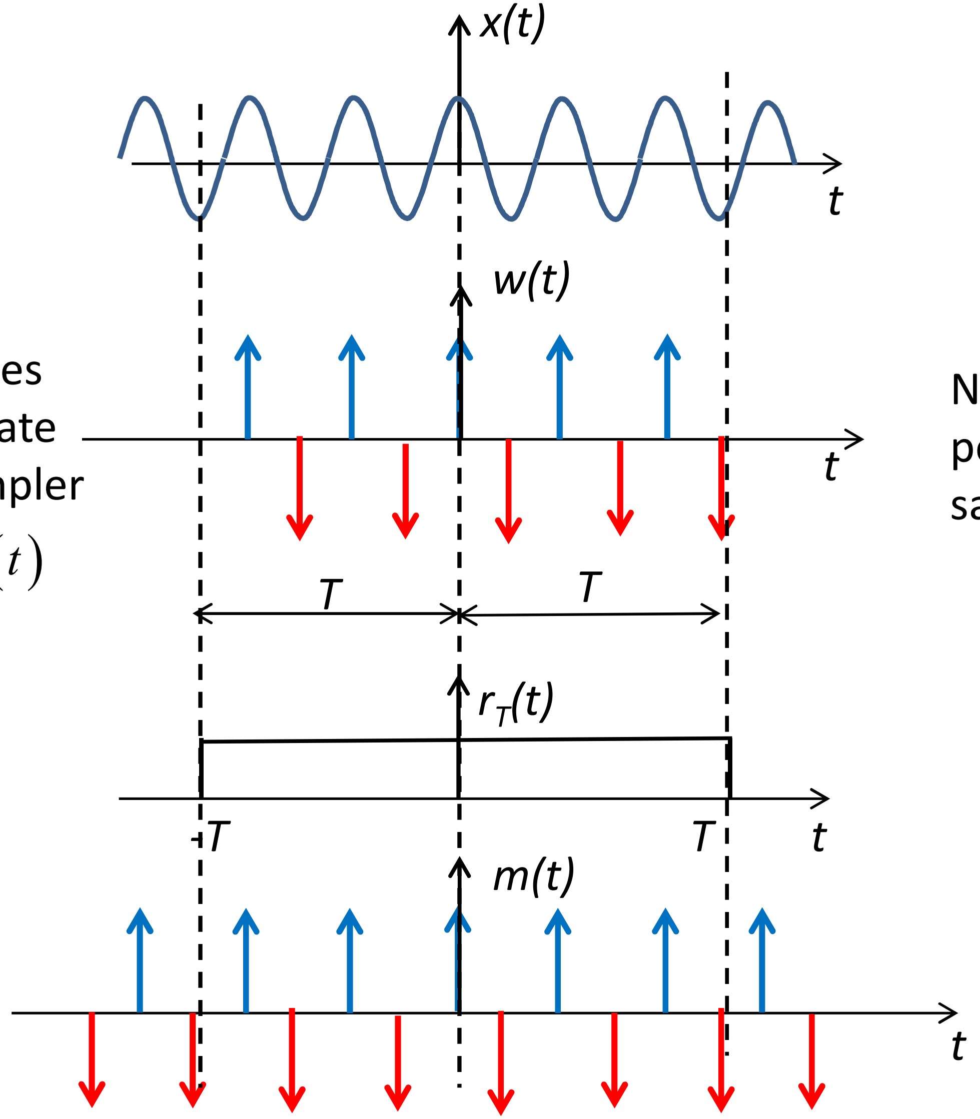
FILTERING: narrow bands at frequencies $0, f_m, 2f_m, 3f_m$ ....

Poor Noise Filtering by Sample-Averaging



- At f_m **useful** band: it collects the **signal** and some white noise around it
- at f = 0 **VERY HARMFUL** band: it collects **1/f noise and no signal**
- at $2f_m$, $3f_m$, ... **harmful** bands: they collect just white noise without any signal

Synchronous measurement with DC suppression by summing positive peak and subtracting negative peak samples

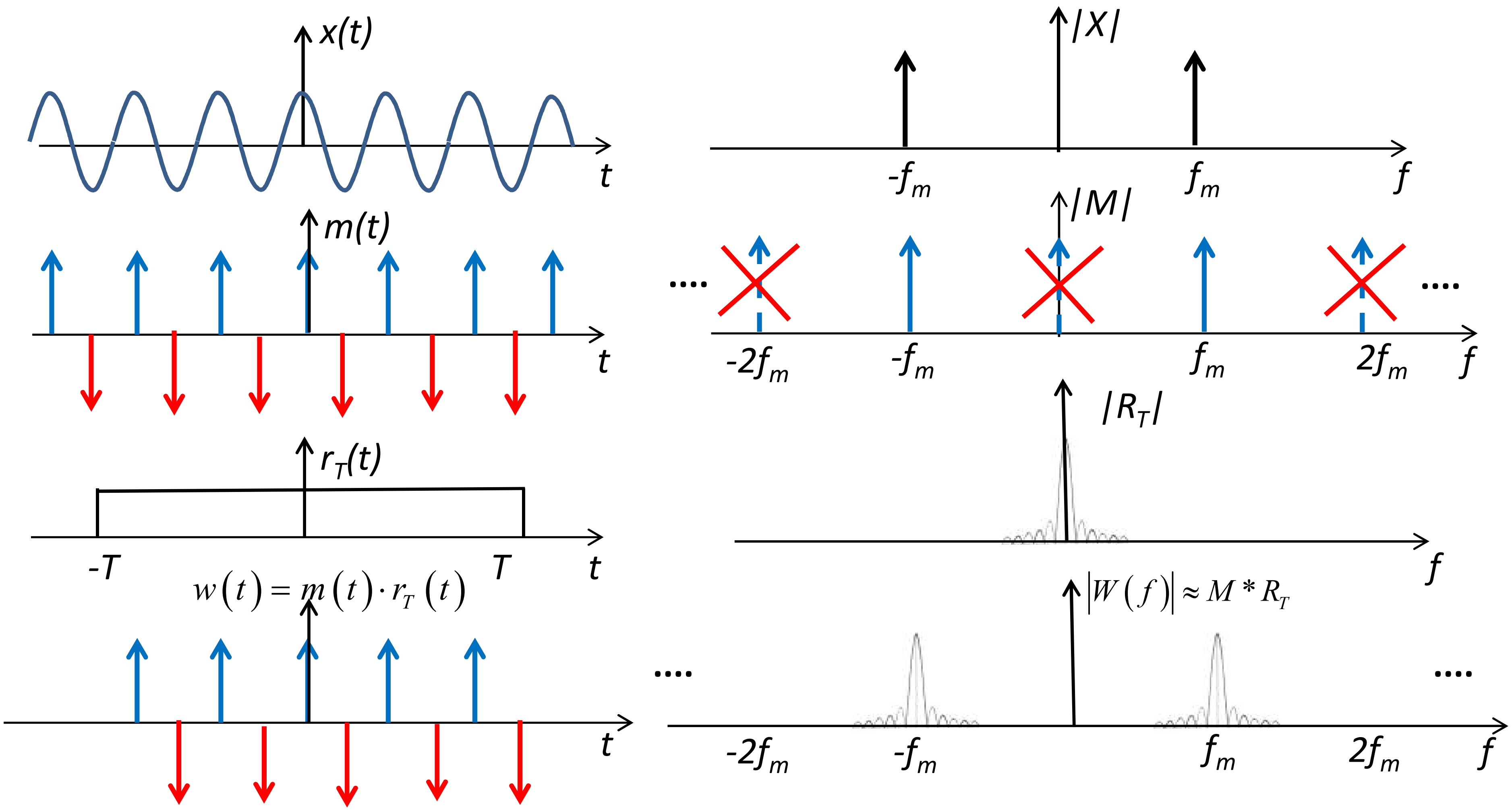


To take 2N samples is equivalent to gate a free-running sampler

$$w(t) = m(t) \cdot r_T(t)$$

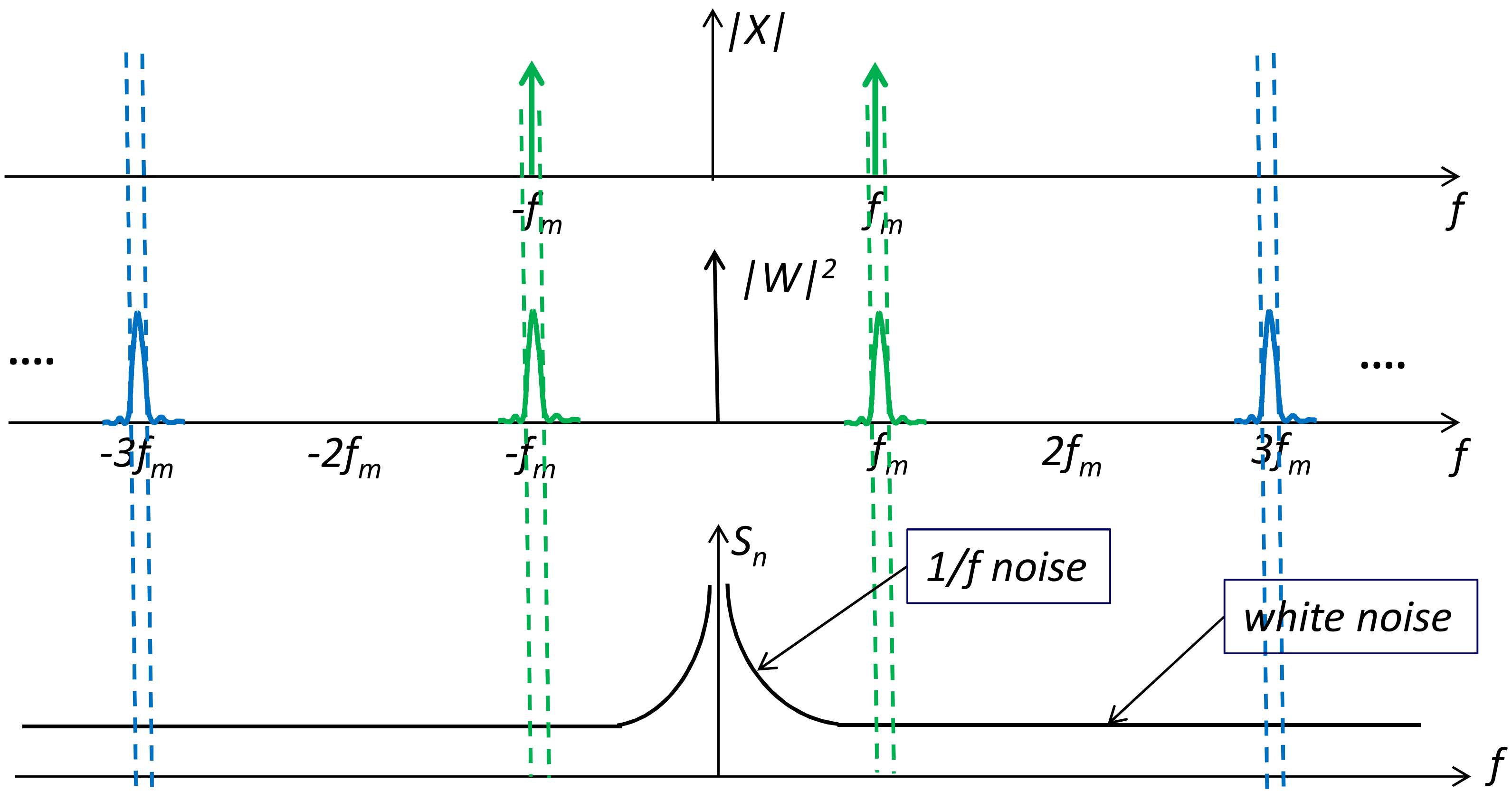
NB: equal number of positive and negative samples (zero net area)

Synchronous measurement with DC suppression by summing positive peak and subtracting negative peak samples



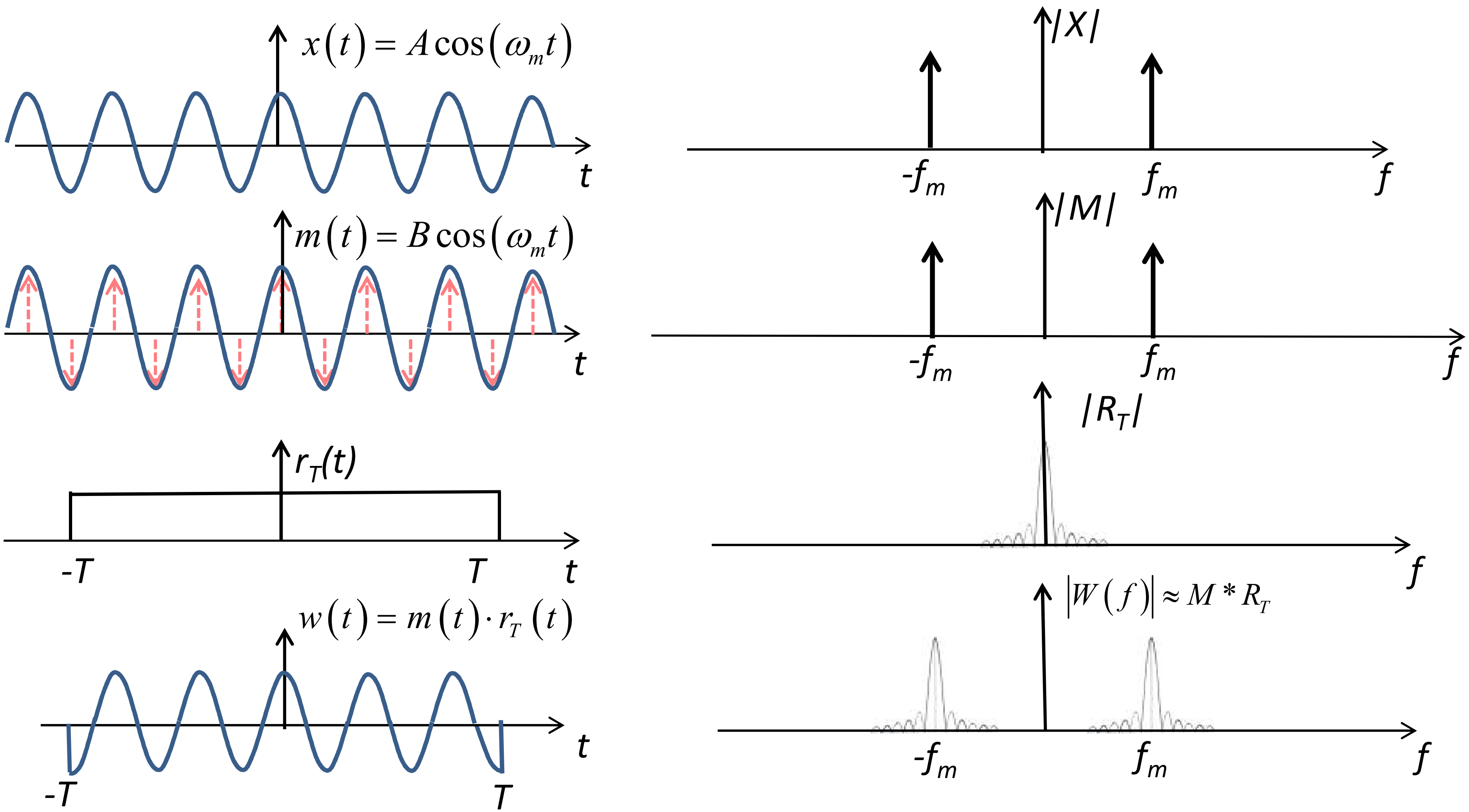
FILTERING : narrow bands at f_m and at **odd** multiples $3f_m$, $5f_m$ ....

Improved Filtering by Sample-Averaging with DC Suppression



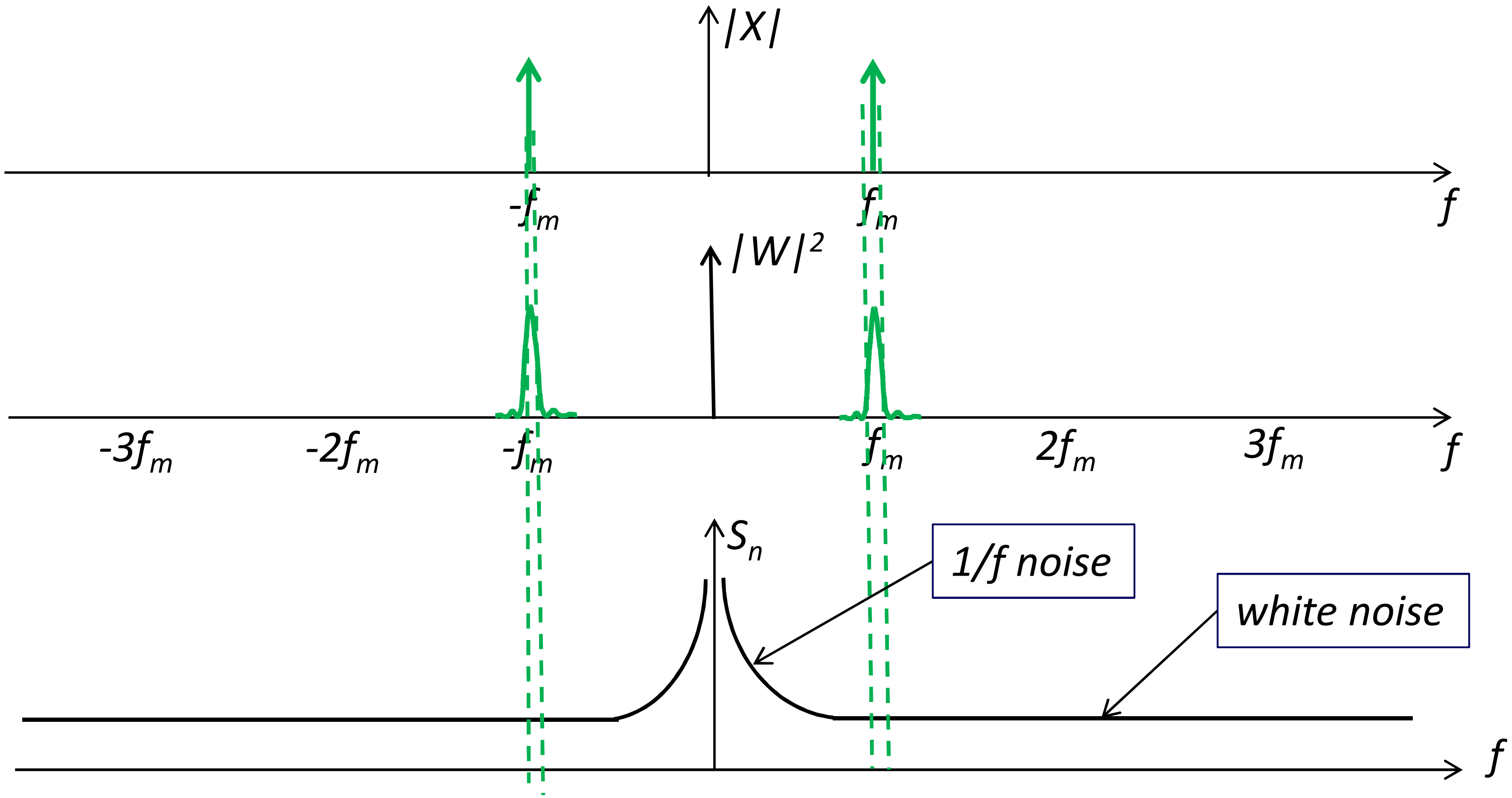
- at f_m **useful** band that collects the **signal** and some white noise around it
- **No more band at f = 0** , no more collection of 1/f noise
- at $3f_m$, $5f_m$ .... residual **harmful** bands that collect just white noise without any signal: how can we get rid also of them?

Continuous sinusoidal weighting
instead of peak sampling



TRULY EFFICIENT FILTERING : just one narrow band at f_m

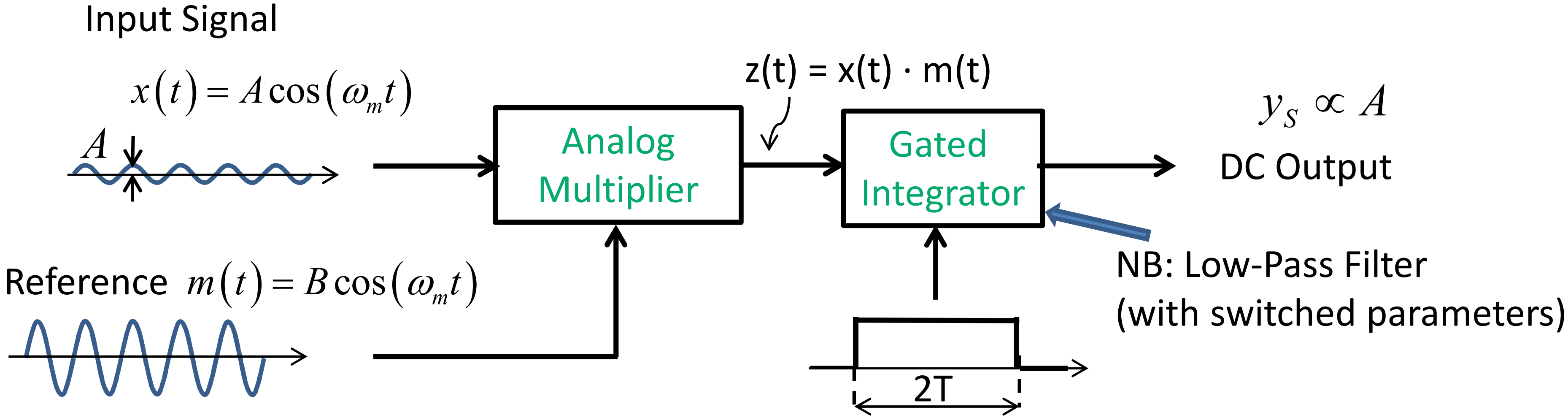
Optimized noise filtering in synchronous measurement



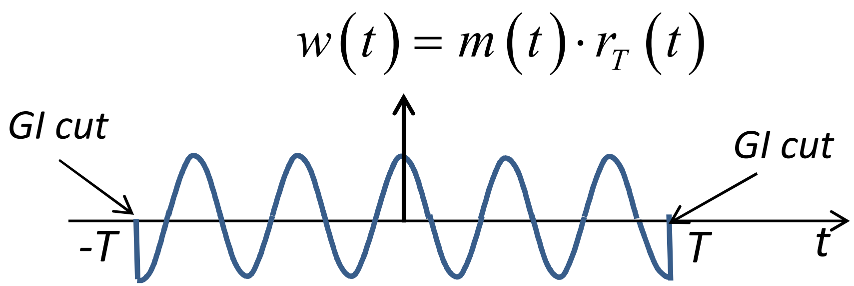
- at f_m **useful** band that collects the **signal** and some white noise around it
- No band at f = 0 , no collection of **1/f noise**
- No residual bands at $3f_m$, $5f_m$ ... no more collection of white noise without any signal

How to implement this optimized synchronous measurement?

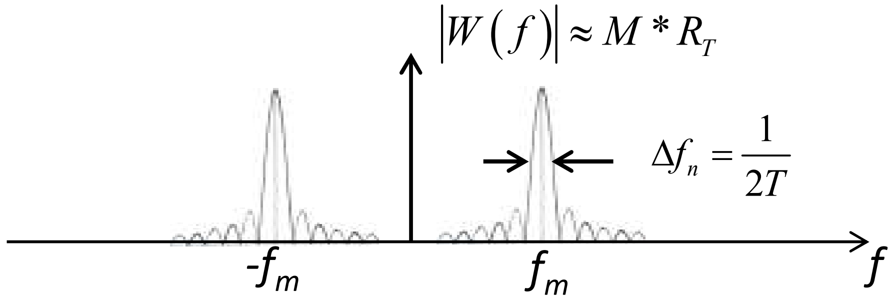
Basic set-up for Synchronous Measurement with optimized noise filtering



Weighting in time domain



Weighting in frequency domain



- **NB** the reference input to the multiplier is a **STANDARD** waveform, which absolutely does **NOT** depend on the signal: it is the same for any signal !!
- Therefore the set-up is a **LINEAR** filter (with time-variant parameters)

Main Advantages of Synchronous Measurements with optimized noise filtering

This linear time-variant filter composed by Analog Multiplier (Demodulator) and Gated Integrator (Low-pass filter with switched-parameter) has a weighting function similar to that of a tuned filter with constant-parameter, but has basic advantages over it:

- Center frequency f_m and width Δf_n are **independently** set
- The **center** frequency is set **by the reference** *m(t)* and locked at the frequency f_m
- In cases where f_m has not a very stable value the filter band-center tracks it: the signal is thus kept in the admission band even if the width Δf_n is very narrow.
- The **width** $\Delta f_n = 1/2T$ is set **by the GI**, it is the (bilateral) passband of the GI
- Narrow Δf_n and high quality factor Q can thus be easily obtained **at any** $\boldsymbol{f_m}$

$$\Delta f_n \ll f_m \qquad\qquad Q = \frac{f_m}{\Delta f_n} \gg 1$$

Intuitive Interpretation of Synchronous Measurement with optimized noise filtering

It is instructive to follow the action of the filter through the various stages, that is

- Multiplier stage, where the reference modulates the input signal, thus shifting the signal in frequency down by $-f_m$ and up by $+f_m$
- Gated Integrator stage, which integrates over 2T the output of the multiplier

By noting that

the input signal $x(t) = A(t)\cos(2\pi f_m t)$ is a modulated signal with A(t) slow with respect to f_m (i.e. with Fourier components *A(f)* only at low frequencies $f << f_m$)

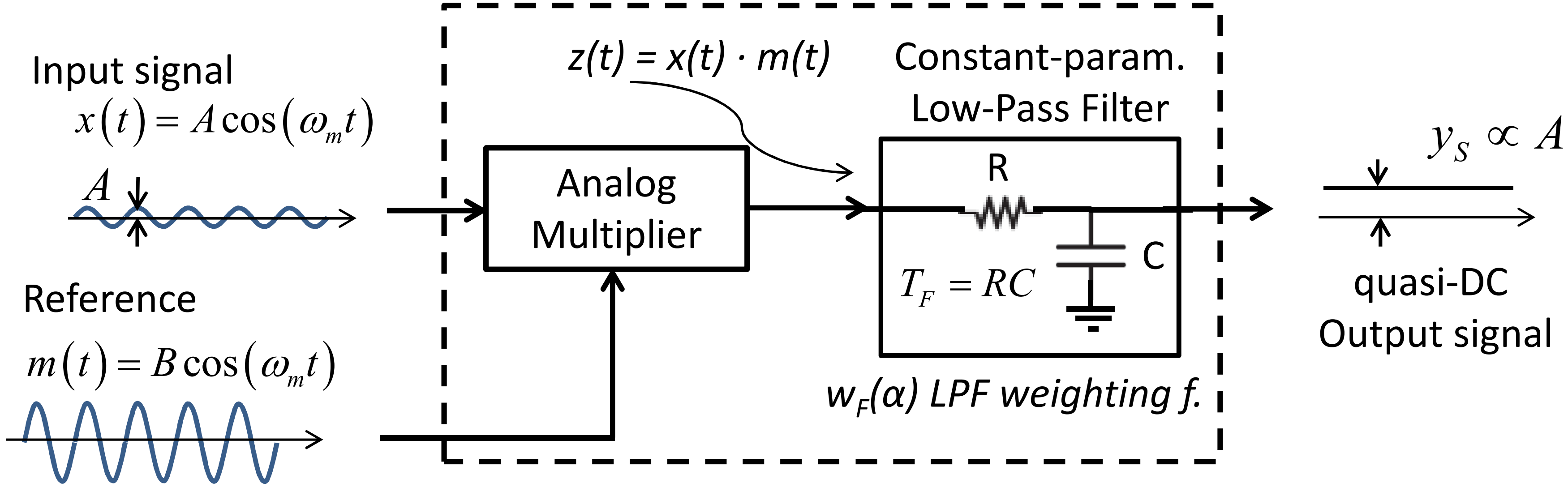
we understand that

- the multiplier performs a **DEMODULATION** that
  a) shifts x(t) in frequency down to $f=0$, that is, brings **A(t) back to its «BASE BAND»**
  b) shifts x(t) also up to the higher frequency $2f_m$
- the Gated Integrator
  1) performs a measurement of A(t) averaged over 2T
  2) cuts off the components with frequency higher than the GI low-pass band-limit

Lock-in Amplifier
Principle and Weighting Function

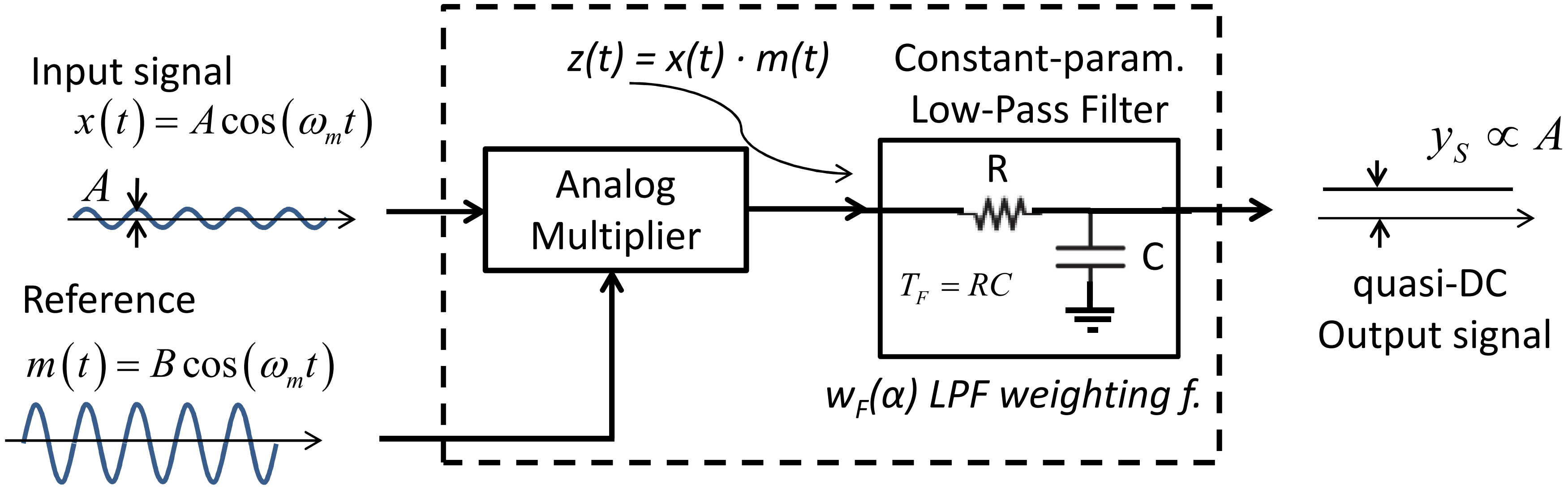
From Discrete to Continuous Synchronous Measurements: principle of the Lock-in Amplifier (LIA)

With averaging performed by a **gated integrator**, the amplitude A can be measured only at **discrete times** (spaced by at least the averaging time 2T). However, by employing a **constant-parameter low-pass** filter instead of the GI, **continuous monitoring** of the slowly varying amplitude A(t) is obtained.



The **constant** parameter LPF performs a **running** average of the output *z(t)* of the demodulator. The output is continously updated and tracks the slowly varying amplitude *A(t)*. This basic set-up is denoted Phase-Sensitive Detector (PSD) and is the core of the instrument currently called **Lock-in Amplifier**.

Principle of the Lock-in Amplifier (LIA)



The **constant** parameter LPF performs a **running** average of *z(t)* over a few T_F that continously updates the output

$$y(t) = \int_0^\infty z(\alpha) w_F(\alpha) d\alpha = \int_0^\infty x(\alpha) m(\alpha) w_F(\alpha) d\alpha$$

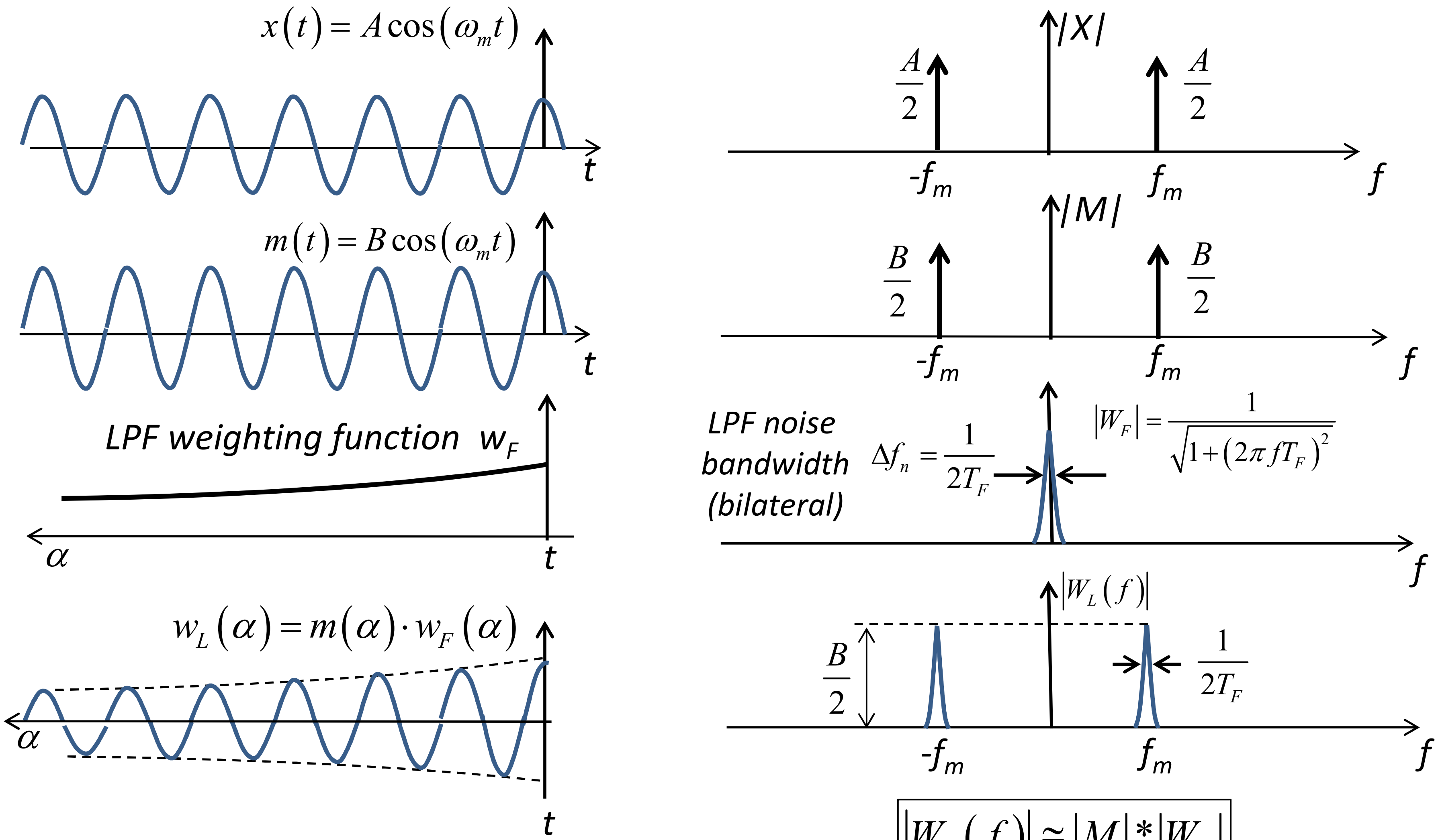
By comparison with the definition of the LIA weighting function $w_L(\alpha)$

$$y(t) = \int_0^\infty x(\alpha) w_L(\alpha) d\alpha$$

we see how the **demodulation** and LPF are combined in the LIA

$$w_L(\alpha) = m(\alpha) \cdot w_F(\alpha) \quad \rightleftarrows \quad |W_L(f)| \cong |M| * |W_F|$$

Weighting Function w_L of the Lock-in Amplifier



$$|W_L(f)| \cong |M| * |W_F|$$

S/N of the Lock-in Amplifier

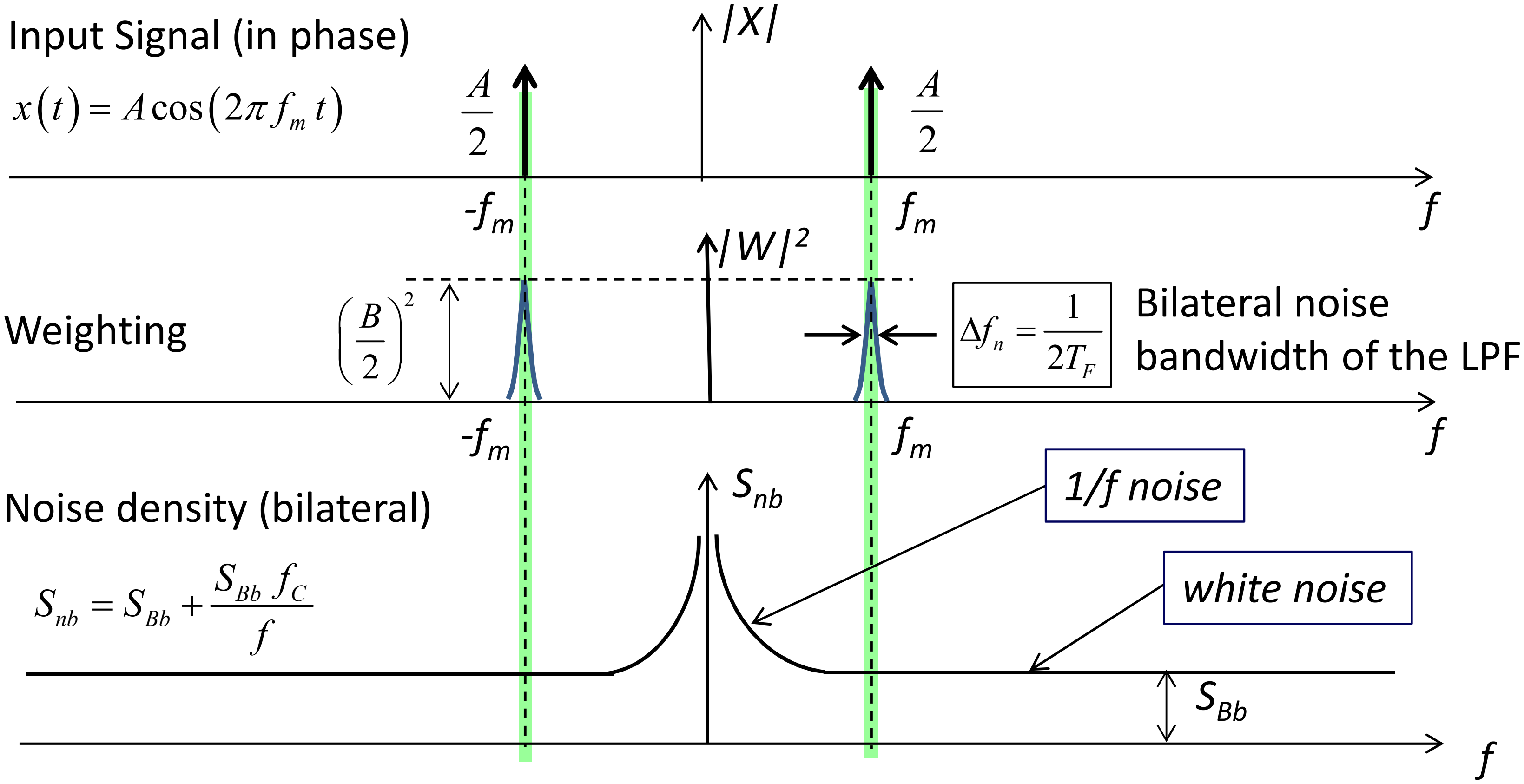
Input Signal (in phase)

$x(t) = A\cos(2\pi f_m t)$

Weighting

Noise density (bilateral)

$S_{nb} = S_{Bb} + \frac{S_{Bb} f_C}{f}$



Output signal $y_S = 2\frac{A}{2}\cdot\frac{B}{2} = \frac{B}{2}A$

Output Noise $\overline{n_{yL}^2} = 2\left(\frac{B}{2}\right)^2 \cdot S_{Bb}\cdot\Delta f_n = \frac{B^2}{2}\cdot S_{Bb}\cdot\Delta f_n$

$$\left(\frac{S}{N}\right)_L = \frac{y_S}{\sqrt{\overline{n_{yL}^2}}} = \frac{A}{\sqrt{2S_{Bb}\Delta f_n}}$$

S/N of the Lock-in Amplifier

$$\left(\frac{S}{N}\right)_L = \frac{A}{\sqrt{2S_{Bb}\Delta f_n}}$$

or in power terms
$$\left(\frac{S}{N}\right)_L^2 = \frac{A^2/2}{S_{Bb}\Delta f_n} = \frac{in - phase\ signal\ power}{half\ power\ of\ white\ noise\ in\ the\ band\ \Delta f_n}$$

S/N equation in terms of the unilateral parameters

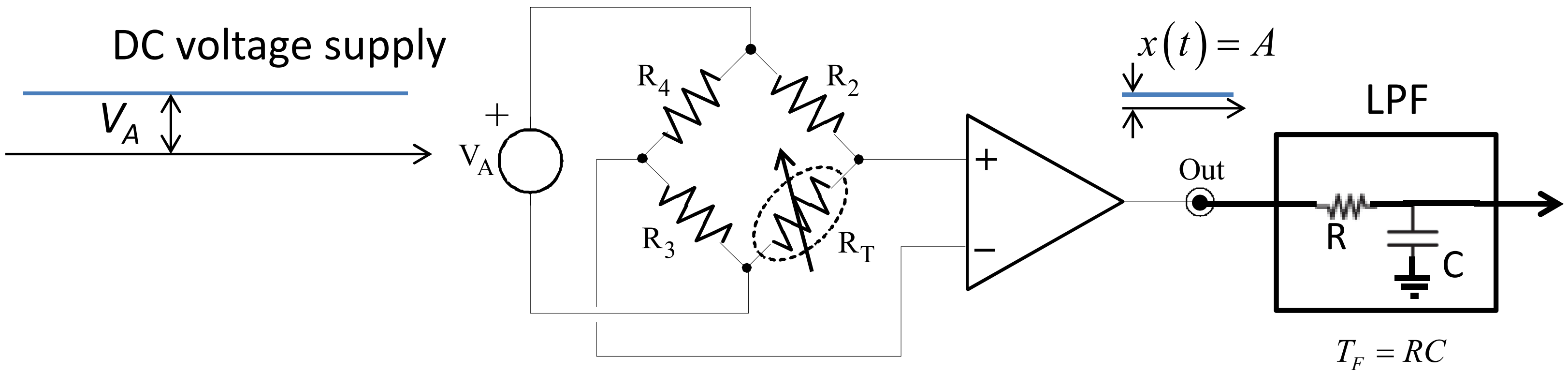
By introducing

- f_{Fn} the LPF unilateral bandwidth (upper band-limit for noise), i.e. $\Delta f_n = 2 f_{Fn}$
- S_{bu} the unilateral noise density, i.e. $2S_{Bb} = S_{Bu}$

we can write

$$\left(\frac{S}{N}\right)_L = \frac{A}{\sqrt{2S_{Bu} f_{Fn}}}$$

Case of DC signal with LPF compared to AC signal with LIA



Let us consider the set-up of the key example (measurement with resistive sensor) now with DC supply voltage V_A equal to the amplitude of the previous AC supply. The signal now is a DC voltage equal to the amplitude A of the previous AC signal.

With a LPF equal to that employed in the previous LIA we obtain:

Output signal $y_C = A$

Output Noise $\overline{n_{yC}^2} = \widehat{S_{nu}} \cdot f_{Fn}$

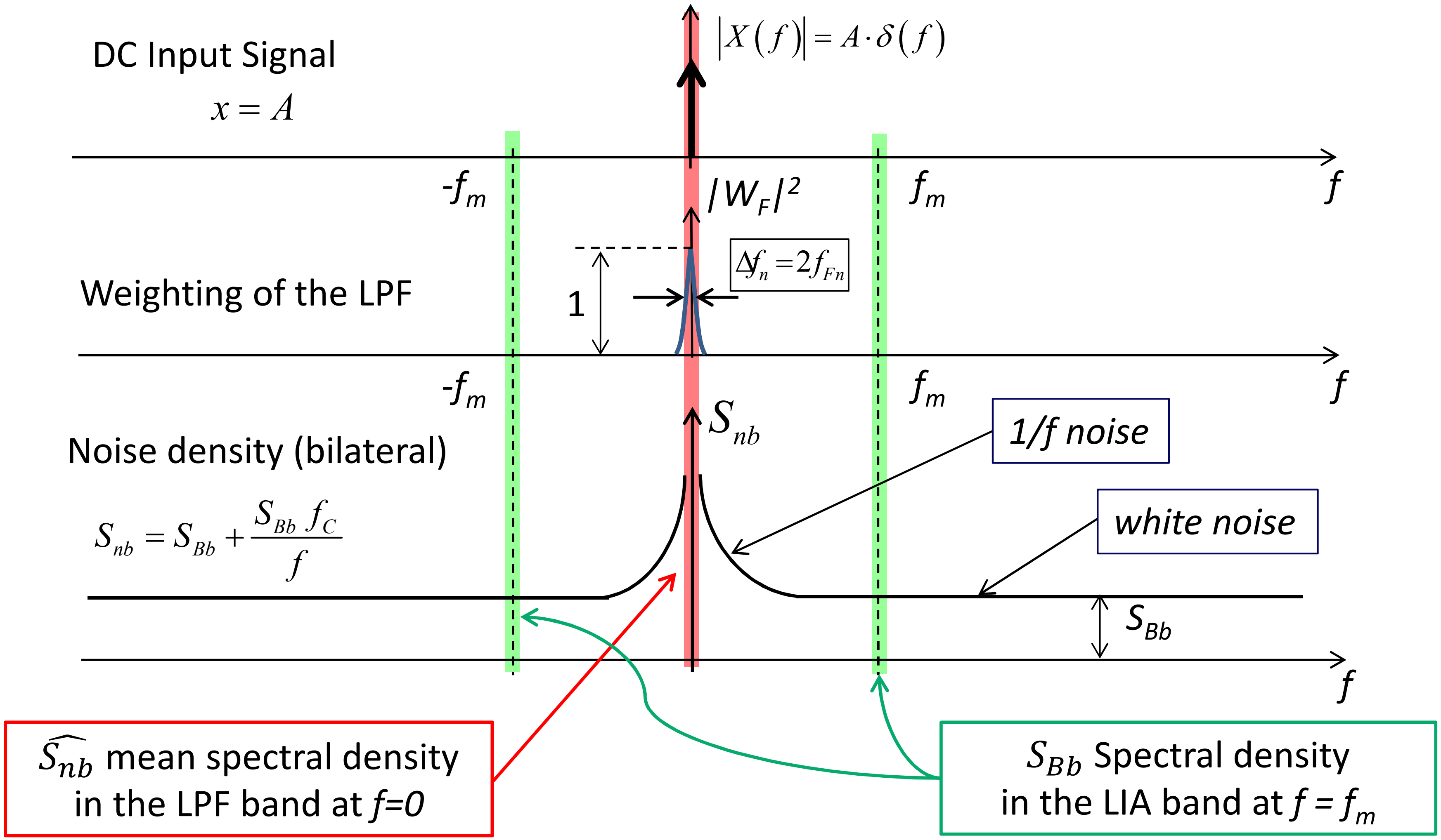
($\widehat{S_{nu}}$ mean density in the LPF band)

$$\left(\frac{S}{N}\right)_C = \frac{y_C}{\sqrt{\overline{n_{yC}^2}}} = \frac{A}{\sqrt{\widehat{S_{nu}} f_{Fn}}}$$

This S/N may look **better by the factor $\sqrt{2}$** than the S/N obtained with the LIA, but is this conclusion true?

NO, such a conclusion is **grossly wrong because $\widehat{S_{nu}} \gg S_{Bu}$!!**

DC signal and LPF compared to AC signal and LIA



A passband at $f = 0$ is a risk: 1/f noise gives $\widehat{S_{nb}} \gg S_{Bb}$!!

Fake LIA passbands arise from imperfect modulation

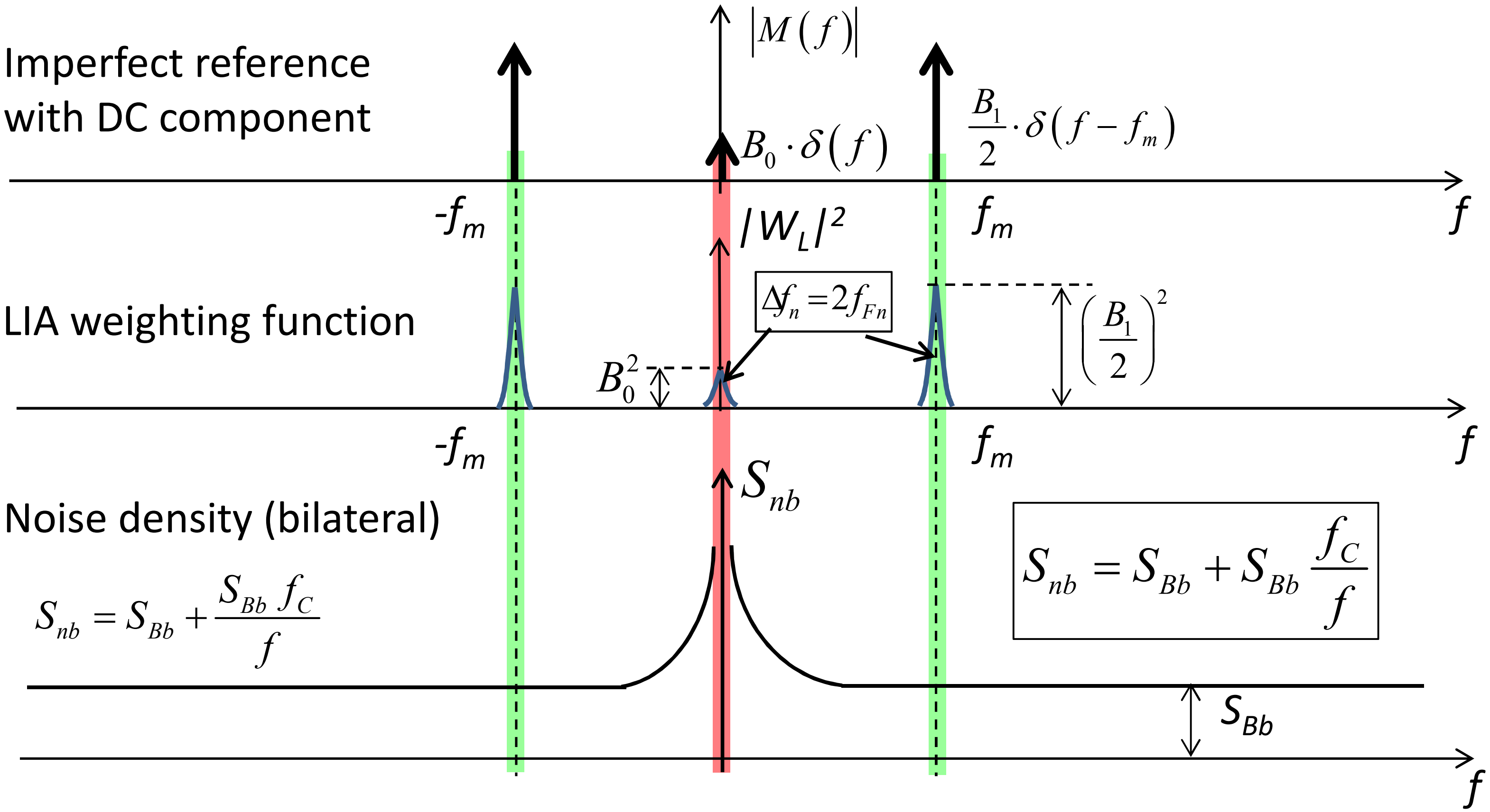
- Ideally, the reference waveform should be a perfect sinusoid at frequency f_m with amplitude B_1
- In reality, deviations from the ideal can generate spurious harmonics at multiples kf_m $(k = 0, 1, 2 ...)$ with amplitudes B_k (small $B_k << B_1$ in case of small deviations)
- Moreover, effects equivalent to an imperfect reference waveform can be caused by non-ideal operation (non-linearity) of the multiplier
- Since it is

$$\left|W_L(f)\right| \cong M(f) * W_F(f)$$

each spurious harmonic component of $M(f)$ adds to the LIA weighting function W_L a spurious passband at frequency kf_m with amplitude B_k and shape given by the LPF

- A fake passband **at $f = 0$ is particularly detrimental even with small $B_0 << B_1$** because it covers the high spectral density of $1/f$ noise ....
- .... and unluckily any **deviation from perfect balance of positive and negative areas** of the reference produces a DC component with associated passband at $f = 0$!!

Fake LIA passband at f = 0

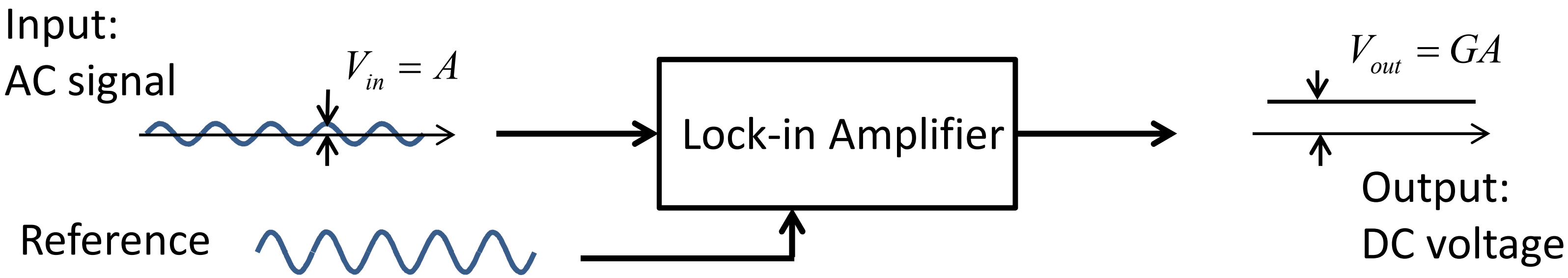


The ratio $\widehat{S_{nu}} / S_{Bu} > f_C / f_{Fn} \gg 1$ can match or exceed the amplitude ratio $B_1^2 / 2B_0^2$

so that the noise in the fake passband $\overline{n_{yL0}^2} = B_0^2 \cdot \widehat{S_{nb}} \cdot \Delta f_n = B_0^2 \cdot \widehat{S_{nu}} \cdot f_{Fn}$

can equal or exceed that in the correct passband $\overline{n_{yL1}^2} = \frac{B_1^2}{2} \cdot S_{Bb} \cdot \Delta f_n = \frac{B_1^2}{2} \cdot S_{Bu} \cdot f_{Fn}$

About the «Gain» of the Lock-in Amplifier



LIA Gain $$G = \frac{output\ voltage}{input\ voltage} = \frac{V_{out}}{V_{in}}$$ however:
- V_{out} is a **DC** voltage
- V_{in} is the amplitude of an **AC** voltage

G is **different in principle** from the gain of an ordinary amplifier: it is a **Transfer Gain** that characterizes the INFORMATION TRANSFER IN FREQUENCY from f_m to DC.

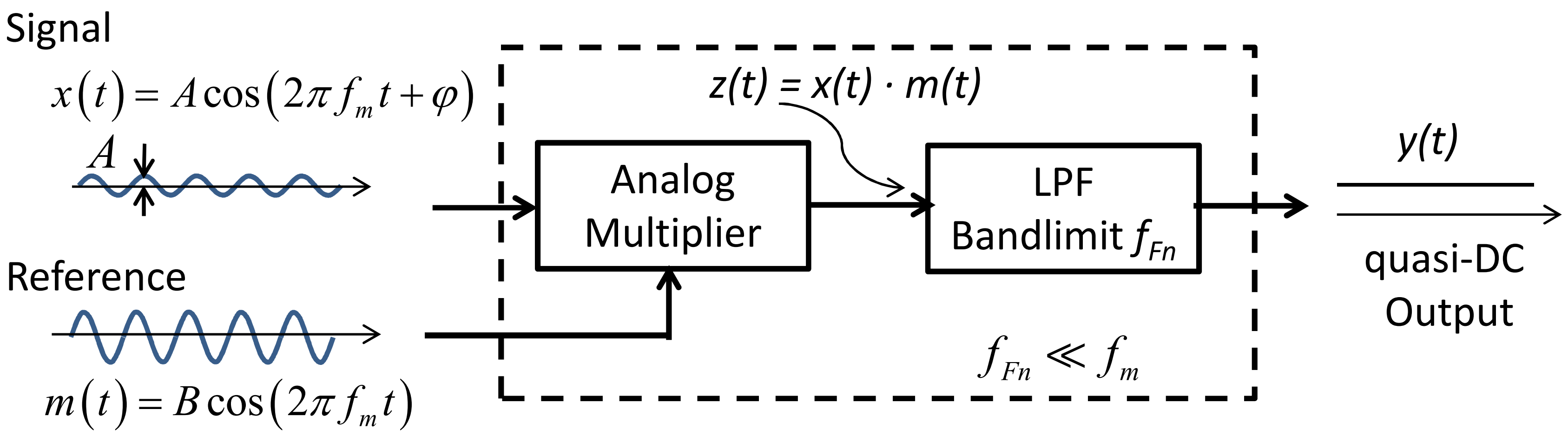
LIA Power Gain $$G_P = \frac{output\ power}{input\ power} = \frac{P_{out}}{P_{in}}$$ however:
- P_{out} is DC power $P_{out} = V_{out}^2$
- P_{in} is AC power $P_{in} = \frac{V_{in}^2}{2}$

Therefore $G_P \neq G^2$ and it is precisely

$$G_P = \frac{V_{out}^2}{\frac{V_{in}^2}{2}} = 2\,G^2$$

Stage-by-Stage view of the Signal in the Lock-in Amplifier

Processing Signals with $f = f_m$ in the Lock-in Amplifier (LIA)



- The Multiplier translates the signal in two replicas shifted in frequency down to $\omega=0$ and up to $\omega=2\omega_m$

$$z(t) = A\cos(\omega_m t + \varphi) B\cos(\omega_m t) = \frac{AB}{2}\cos\varphi + \frac{AB}{2}\cos(2\omega_m t + \varphi)$$

- The LPF (that has cutoff f_{Fn} much lower than the signal frequency f_m) passes to the output practically only the constant component

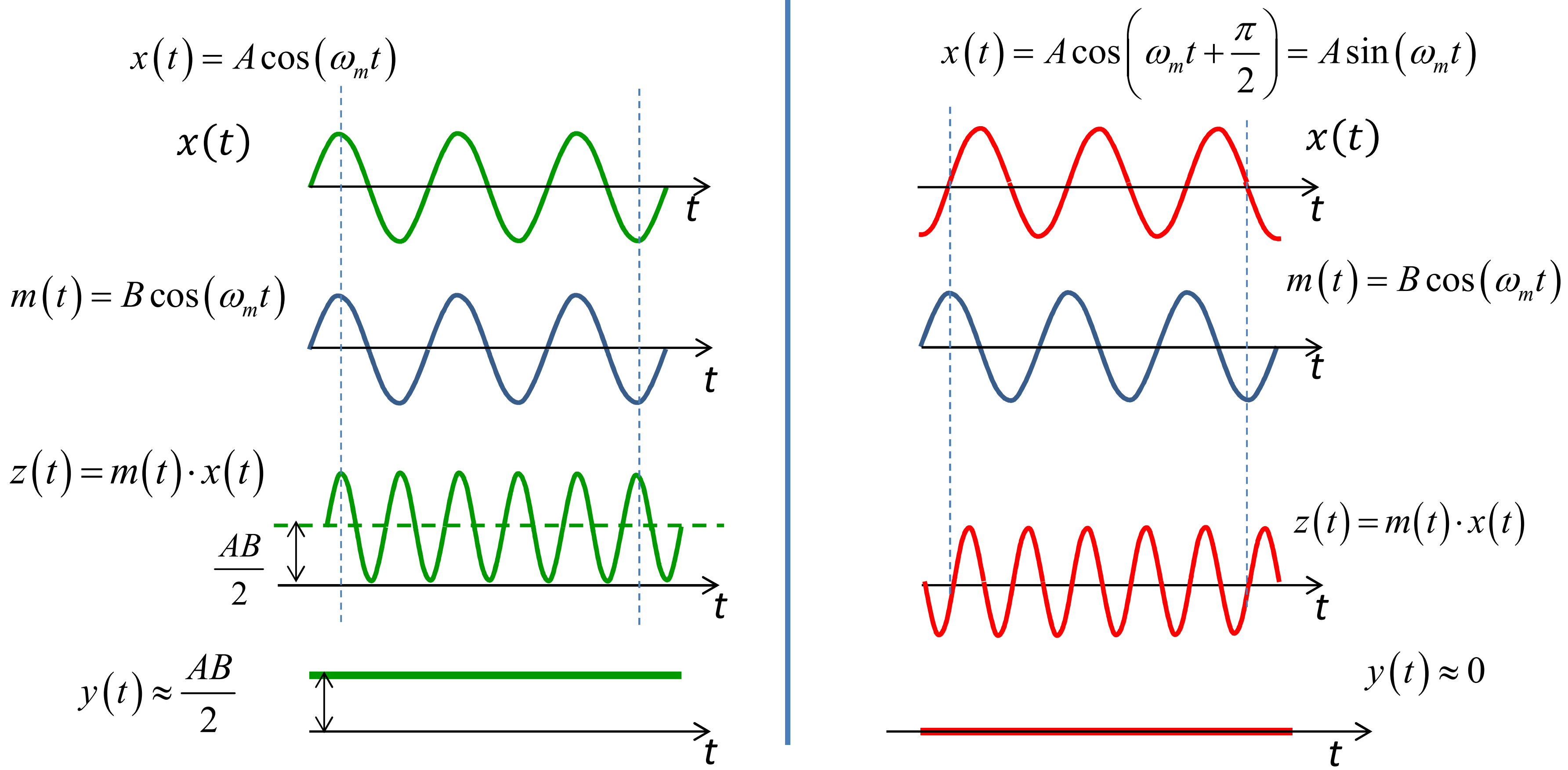
$$y(t) \approx \frac{AB}{2}\cos\varphi$$

We can gain a better insight by considering the special cases of signal that with respect to the reference are **in phase** [$\varphi=0$ or $k\cdot\pi$] and **in quadrature** [$\varphi=\pi/2$ or $(2k+1)\cdot \pi/2$]

Processing Signals **with *f* =*f*$_m$** in the Lock-in Amplifier (LIA)

Signal **in phase** φ=0

Signal **in quadrature** φ = π/2



- Any sinusoid can be analyzed as sum of in-phase and in quadrature components

$$x(t) = A\cos(\omega_m t + \varphi) = (A\cos\varphi)\cdot\cos\omega_m t - (A\sin\varphi)\cdot\sin\omega_m t$$

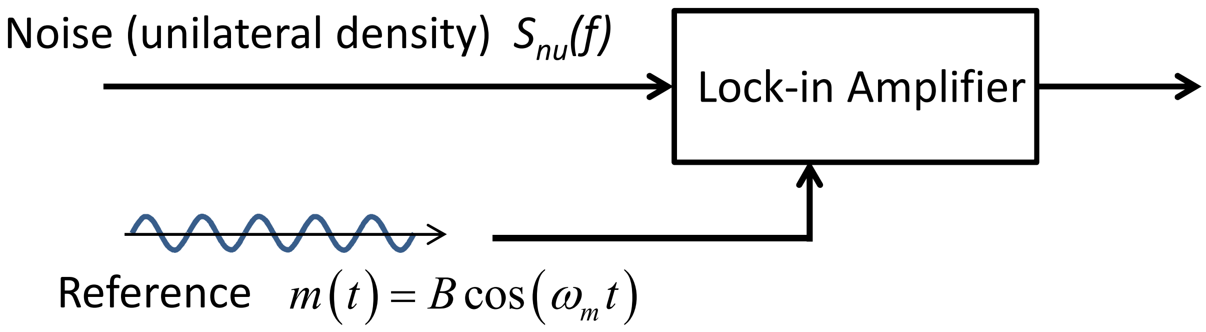
- **Phase selection** by the LIA: the **quadrature** components **do not contribute** to the output

Processing Signals **with $f_s \neq f_m$** in the Lock-in Amplifier (LIA)

- The multiplier converts the sinusoidal signal of frequency $f_s \neq f_m$ in:
  a high-frequency component at $f_h = f_s + f_m$
  plus a low-frequency component at $f_l = |f_s - f_m|$
- The high-frequency component is cut off by the LPF with bandlimit $f_{Fn} << f_m < f_s + f_m$
- The low-frequency component is first subject to selection in frequency and is admitted only if f_l is within the LPF passband (i.e. if $|f_s - f_m| < f_{Fn}$), otherwise it is (approximately) cut-off.
- A signal with frequency f_s admitted by the frequency selection is subjected to further **selection in phase:** only its in-phase component (φ=0) contributes to the output.
- In summary: a sinusoidal signal at frequency $f_s \neq f_m$
  - if f_s is in the **admission bandwidth $\Delta f_n = 2 f_{Fn}$** centered at f_m (i.e. $|f_s - f_m| < f_{Fn}$) the signal is processed by the LIA just like a signal at $f_s = f_m$
  - if f_s is out of the admission bandwidth, it is cut off

Stage-by-Stage view of Noise in the Lock-in Amplifier

Noise in the Lock-in Amplifier (LIA)



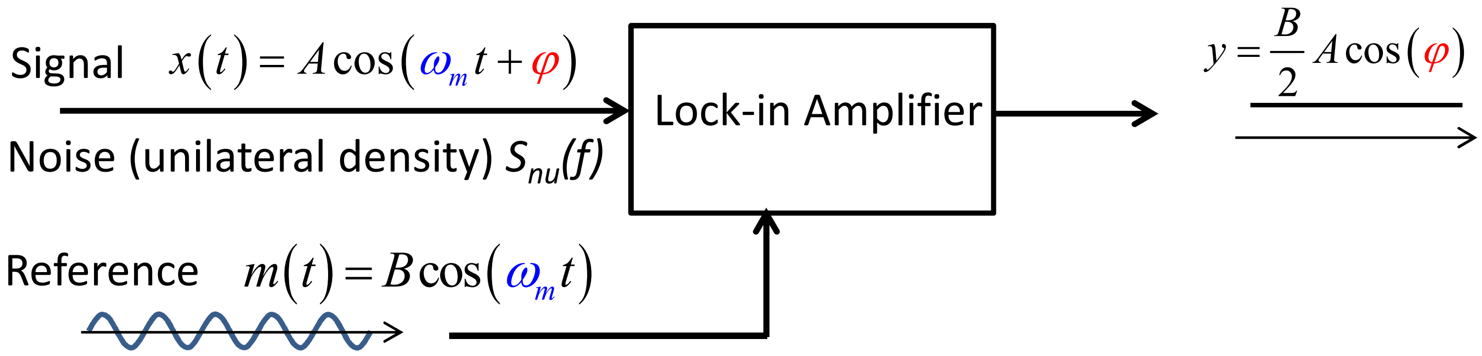
- **Selection in frequency**: noise components are admitted only from $(f_m - f_{Fn})$ to $(f_m + f_{Fn})$, that is, within a bandwidth $\Delta f_n = 2 f_{Fn}$ set by the LPF and centered at f_m . The **total** mean power of the input noise $S_{nu}(f)$ in this admission band is

$$\overline{n^2} = S_{nu}(f_m)\Delta f = S_{Bu}(f_m)\cdot 2 f_{Fn}$$

- **Selection in phase**: only components in phase with the reference contribute to the output. The noise components have **random phase φ , with uniform probability** for all phases from φ=0 to φ=2π. In the ensemble no phase is favoured and the mean power is equally split between in-phase and quadrature components: only the **in-phase** noise mean power is transmitted to the output and is half of the total

$$\overline{n_o^2} = G_P \cdot \frac{1}{2} \cdot \overline{n^2} = \frac{B^2}{2} \cdot S_{Bu}(f_m) f_{Fn}$$

S/N of the Lock-in Amplifier (LIA)



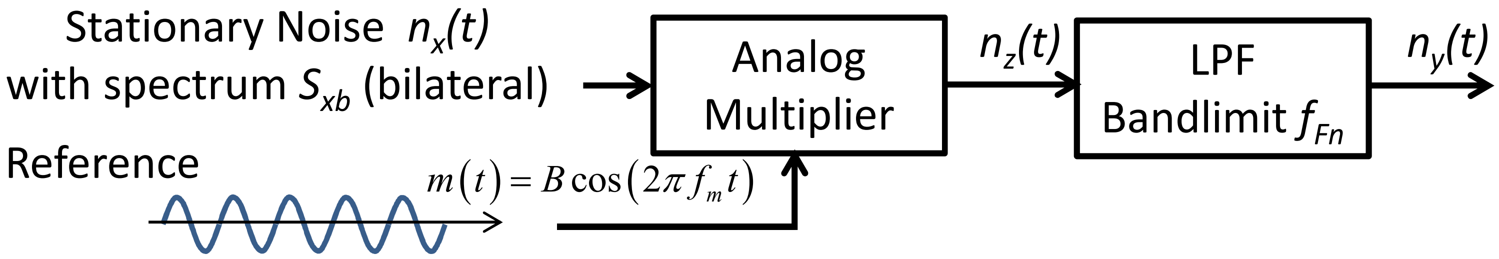
Output signal power $P_y = G_P \frac{\left[A\cos(\varphi)\right]^2}{2} = \frac{B^2}{2} \cdot \frac{1}{2}\left[A\cos(\varphi)\right]^2$

Output noise power $\overline{n_o^2} = G_P \cdot S_{nu}(f_m) f_{Fn} = \frac{B^2}{2} \cdot S_{Bu}(f_m) f_{Fn}$

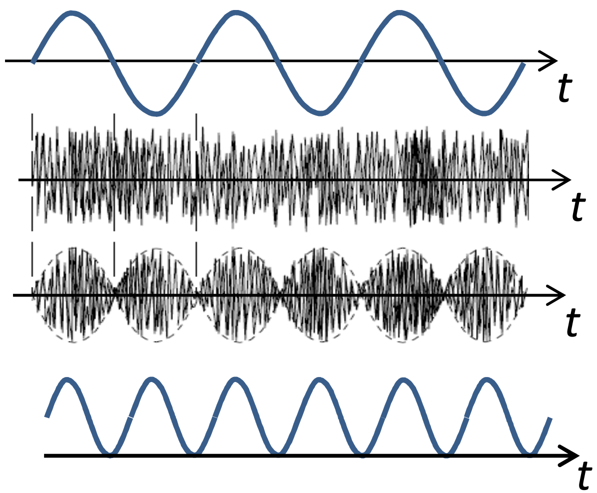
Signal-to-Noise Ratio: it is confirmed that

$$\left(\frac{S}{N}\right)^2 = \frac{(A\cos\varphi)^2}{2S_{Bu}(f_m) f_{Fn}} \Longrightarrow \boxed{\left(\frac{S}{N}\right) = \frac{A\cos\varphi}{\sqrt{2S_{Bu}(f_m) f_{Fn}}}}$$

Notice about the noise transfer in frequency



We might think that the **noise spectrum at the multiplier output** is simply the superposition of two replicas of the **input spectrum shifted up and down by f_m**, but this is not exactly the case. The input noise $n_x(t)$ is stationary, whereas the multiplier output noise $n_z(t)$ is **not stationary,** it has cyclically varying intensity so that it is denoted «**cyclostationary**» noise



$m(t) = B\cos(\omega_m t)$ Reference waveform

$n_x(t)$ input noise waveform

$n_z(t) = m(t)\, n_x(t)$ multiplier out noise waveform

$\overline{n_z^2(t)} = \overline{n_x^2} B^2 \cos^2(\omega_m t)$ multiplier out noise power

However, we will show that the LIA **output noise power** can be computed with an equivalent **stationary** noise, namely with the **average in time** of the cyclostationary noise

Notice about the noise transfer in frequency

At the **multiplier out**, the noise autocorrelation function is **not stationary,** but cyclic

$$R_{zz}(t,t+\gamma) = \overline{z(t)z(t+\gamma)} = m(t)m(t+\gamma)\overline{x(t)\cdot x(t+\gamma)} = m(t)m(t+\gamma)\cdot R_{xx}(t,t+\gamma)$$

Since the input is stationary $R_{xx}(t,t+\gamma) = R_{xx}(\gamma)$; with $m(t) = B\cos(\omega_m t)$ we get

$$R_{zz}(t,t+\gamma) = \frac{B^2}{2}\left[\cos\gamma + \cos(2\omega_m t+\gamma)\right]\cdot R_{xx}(\gamma)$$

The LPF with $f_{Fn} \ll f_m$ performs then an average over a time much longer than the period of f_m . Therefore, the noise power at the output of the LPF can be computed employing the time-average of the cyclostationary noise n_z . We can thus employ as **equivalent** stationary autocorrelation $R_{zz,eq}(\Upsilon)$ **the time-average of $R_{zz}(t,t+\Upsilon)$**

$$R_{zz,eq}(\gamma) = \left\langle R_{zz}(t,t+\gamma)\right\rangle = \left\langle m(t)m(t+\gamma)\right\rangle\cdot R_{xx}(\gamma) = K_{mm}(\gamma)\cdot R_{xx}(\gamma)$$

In the frequency domain, the Fourier transform of $R_{zz,eq}(\Upsilon)$ can be employed as equivalent stationary noise spectrum

$$S_{z,eq}(f) = S_m(f) * S_{xb}(f)$$

*(NB: the spectral densities here are **BILATERAL**)*

Notice about the noise transfer in frequency

With sinusoidal reference $m(t) = B\cos(\omega_m t)$

$$K_{mm}(\gamma) = \frac{B^2}{2}\cos(\omega_m \gamma) \rightleftarrows S_m(f) = \frac{B^2}{4}\left[\delta(f - f_m) + \delta(f + f_m)\right]$$

we have

$$R_{zz,eq}(\gamma) = \frac{B^2}{2}\cos(\omega_m \gamma)\cdot R_{xx}(\gamma) \rightleftarrows S_{z,eq}(f) = \frac{B^2}{4}\left[S_x(f - f_m) + S_x(f + f_m)\right]$$

Thanks to the averaging action of the LPF, the output power actually can be computed as if the multiplier output were a stationary spectrum, obtained by **shifting** in frequency by $-f_m$ and by $+f_m$ the **input spectrum** and by **superposing** the two replicas**.**

This operation is usually denoted «**spectrum folding**».

Note that the spectrum folding doubles the spectral density at $f=0$

so that

$$S_{z,eq}(0) = \frac{B^2}{4}\left[S_x(-f_m) + S_x(+f_m)\right] = \frac{B^2}{4}\cdot 2S_{Bb}$$

$$\overline{n_o^2} = S_{Bz,eq}(0)\Delta f = \frac{B^2}{4}\cdot 2S_{Bb}\Delta f$$

or, with unilateral parameters

$$\overline{n_o^2} = \frac{B^2}{2}\cdot S_{Bu} f_{Fn}$$

Bandwidth, Response Time and S/N of the Lock-in Amplifier

Limit to narrow bandwidth convenience

By processing a sinusoidal signal of amplitude *A* in presence of noise S_{Bu} a LIA gives

$$\left(\frac{S}{N}\right) = \frac{A}{\sqrt{2 S_{Bu}\left(f_m\right) f_{Fn}}}$$

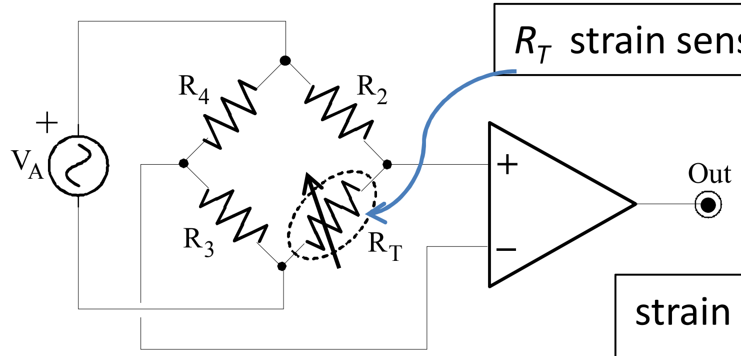
- The equation shows that S/N is improved by reducing the LPF bandlimit f_{Fn} .
- This conclusion, however, is not indefinitely valid in real cases: there is a limit bandwidth, below which the S/N is degraded.
- The reason is that real signals are modulated sinusoids, they have amplitude that varies with time *A(t)* (though very slowly with respect to the signal period $1/f_m$)

$$x(t) = A(t)\cos\left(2\pi f_m t\right)$$

- Let's see how and why this reason limits the S/N improvement obtained by bandwidth reduction. The stage-by-stage view of signal processing readily and intuitively clarifies it.

Real Signals are Modulated Sinusoids

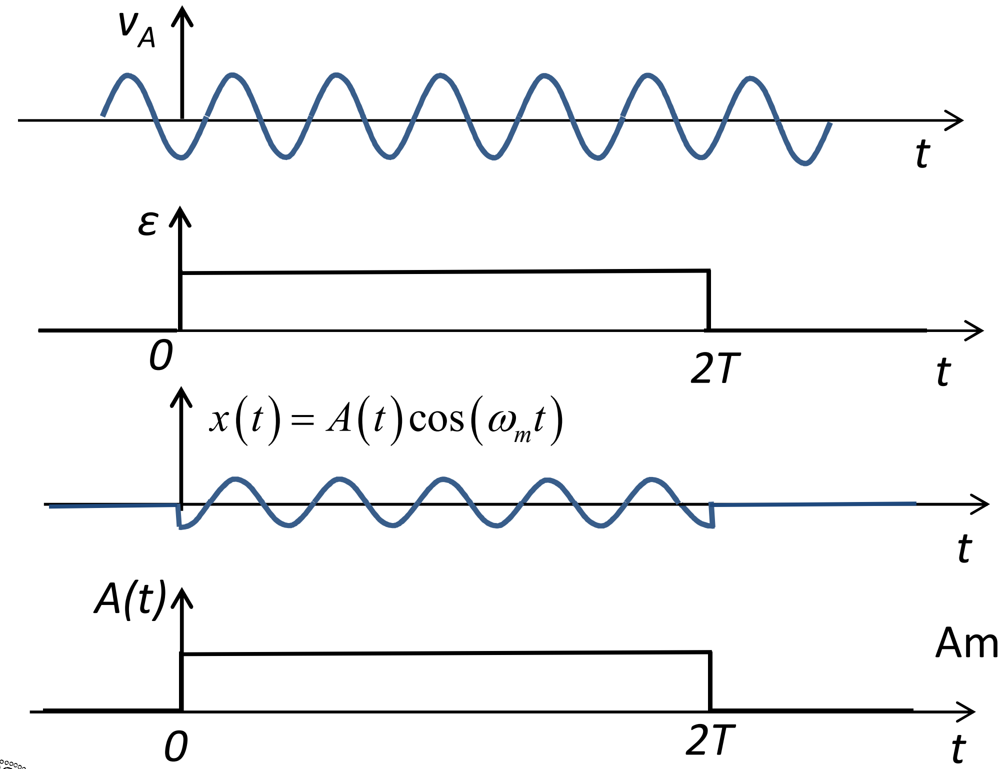
$v_A(t) = V_A \cos(\omega_m t)$



$x(t) = A(t)\cos(\omega_m t)$

R_T strain sensor subjected to variable strain $\varepsilon(t)$

strain ε → variation $\Delta R_T \propto \varepsilon$ →
→ Bridge unbalance → signal $A \propto \varepsilon$



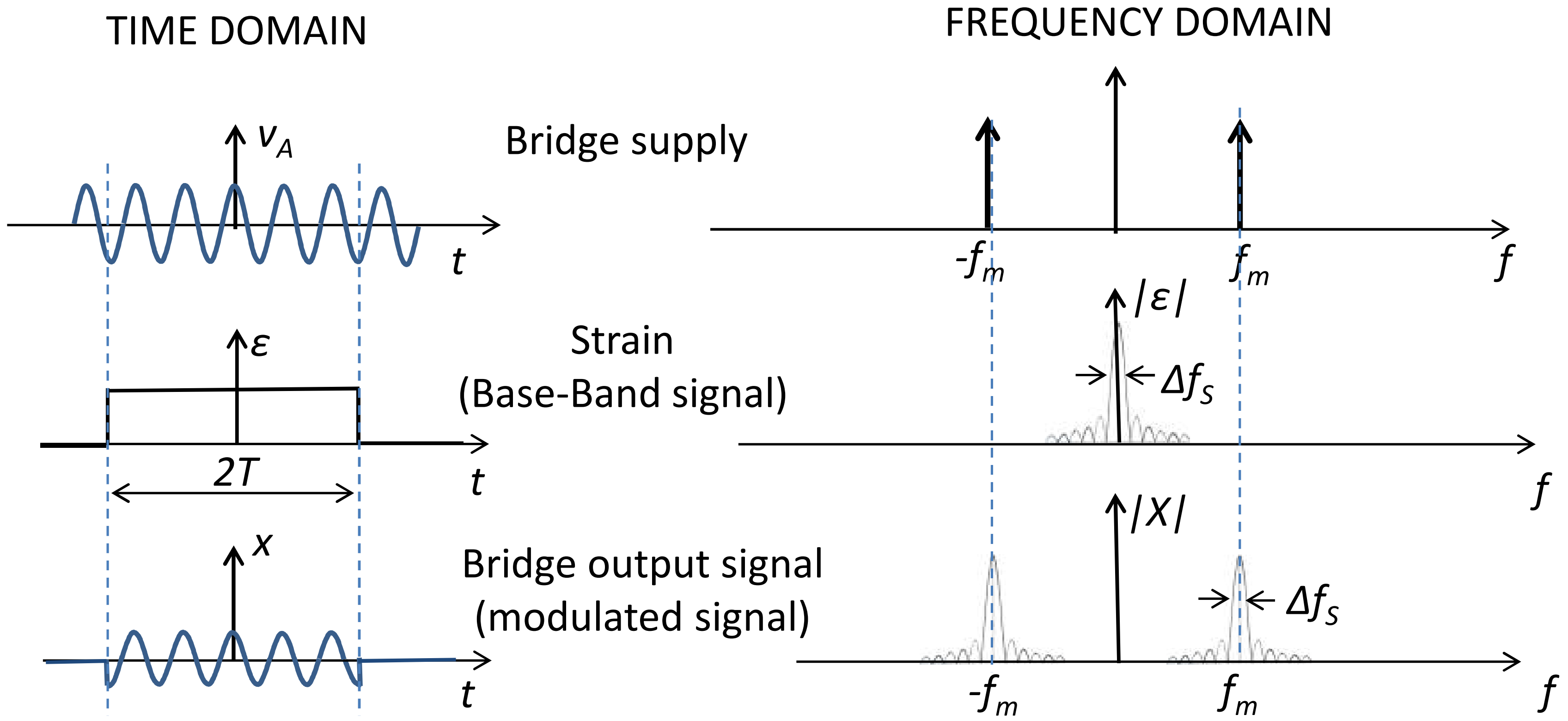
Bridge supply voltage

Typical example of Base Band signal
(signal to be recovered):
Strain ε with finite duration $2T$

Bridge output signal

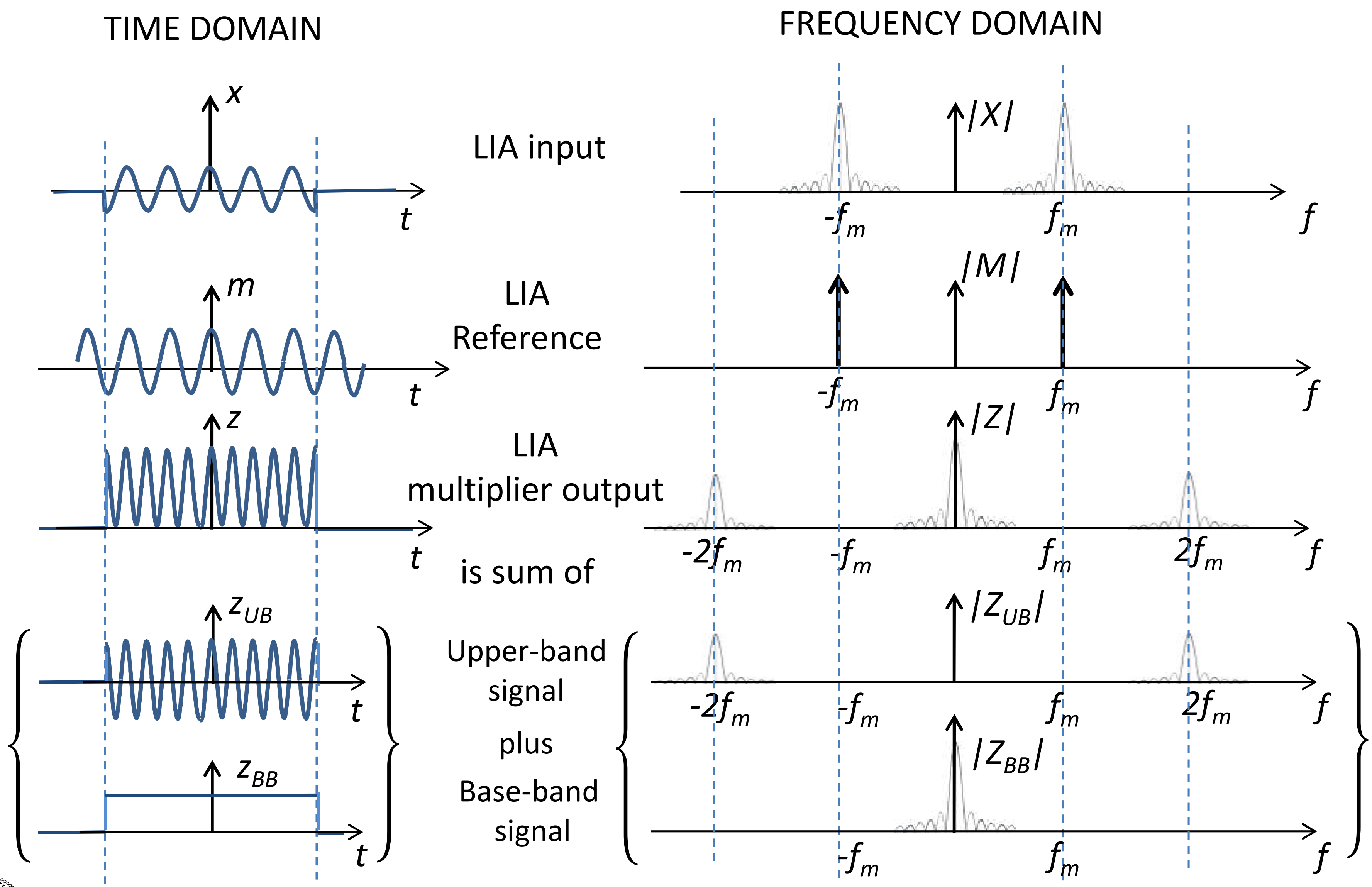
Amplitude modulation = Base Band signal

Real Signals are Modulated Sinusoids



1. The Base-band signal is not constant, it is slowly variable, hence the modulated signal has a **finite bandwidth Δf_S around f_m**
2. in the LIA, the modulated signal is de-modulated by the multiplier
3. the demodulated signal is then filtered by the LPF in the LIA

Signal de-modulation by the multiplier



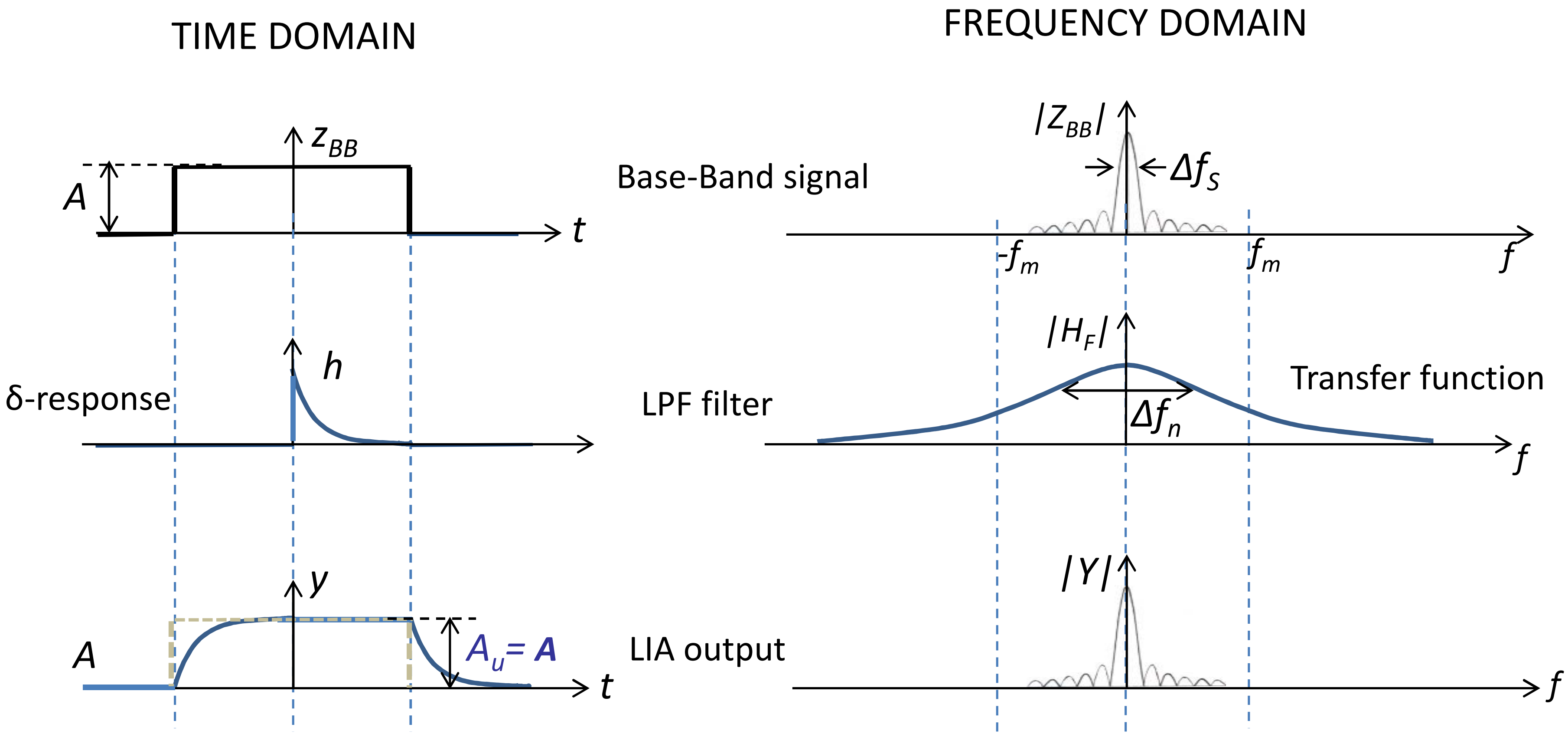
Base-Band signal filtered by the LPF in LIA

The LPF 1) cuts off the Upper-Band signal and
2) filters the Base-Band signal. We may note that:

- amplitude A of Base-Band signal $z_{BB}(t) \propto$ area of $|Z_{BB}(f)| \propto$ its bandwidth Δf_S
- amplitude A_u of LIA output signal $y(t) \propto$ area of $|Y(f)| \propto$ bandwidth of $|Y(f)|$
- LIA output signal $y(t)$ = Base-Band signal $z_{BB}(t)$ filtered by LPF in the LIA
- a) for LPF with band $\Delta f_n >> \Delta f_S$:
  - bandwidth of $|Y(f)|=$ bandwidth Δf_S of BB signal
  - full output signal amplitude $A_u = A$
  - S/N is improved by reducing Δf_n because signal stays constant and rms noise is reduced
- b) for LPF with band $\Delta f_n << \Delta f_S$:
  - bandwidth of $|Y(f)|=$ bandwidth Δf_n of LPF
  - reduced output signal amplitude $A_u \approx A \cdot \Delta f_n / \Delta f_S << A$
  - S/N is degraded by reducing Δf_n : signal decreases as the bandwidth Δf_n whereas rms noise decreases only as the square root of the bandwidth $\sqrt{\Delta f_n}$

Base-Band signal filtered by the LPF in LIA

Case a) LPF band $\Delta f_n >> \Delta f_S$ BB signal band

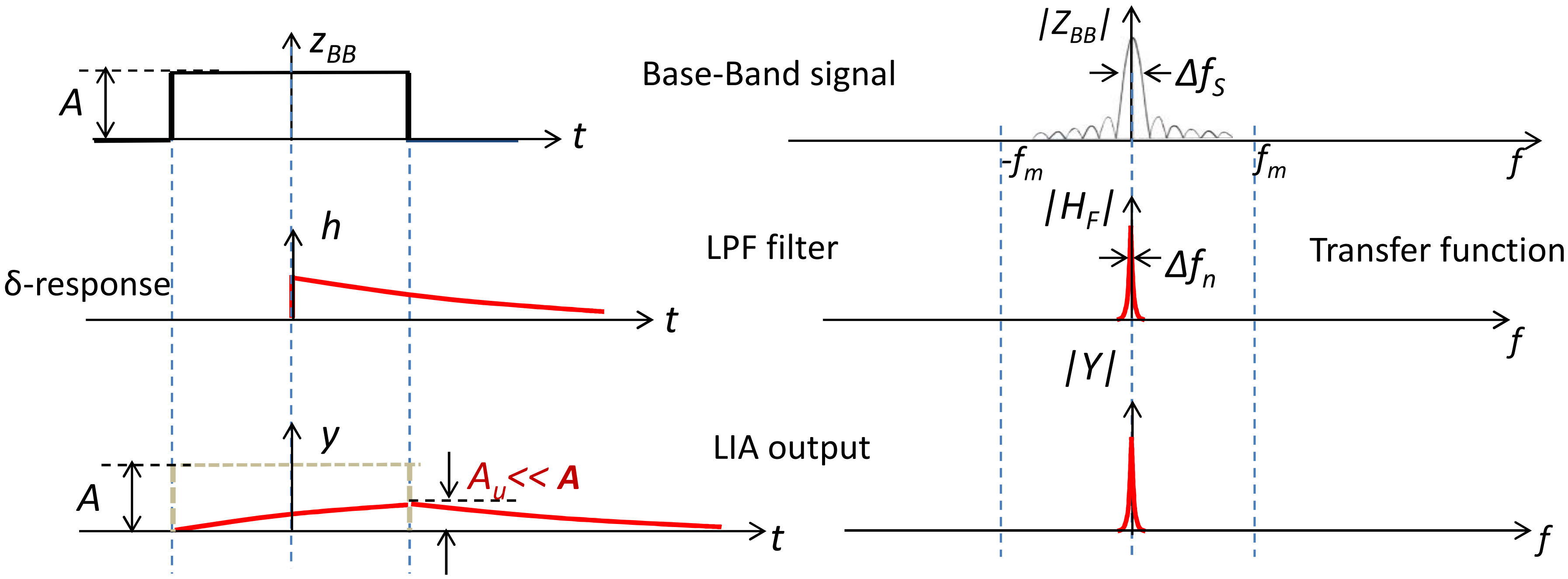


Base-Band signal filtered by the LPF in LIA

Case b) LPF band $\Delta f_n << \Delta f_S$ BB signal band

TIME DOMAIN

FREQUENCY DOMAIN



Time-domain study of the LIA response time

The waveform of the output signal of a LIA can also be computed directly in time.

With LIA input signal $x(\alpha) = a(\alpha) \cdot m(\alpha)$

LIA reference $B \cdot m(\alpha)$

LPF in LIA weighting function $w_F(\alpha)$

we have

$$y(t) = \int_{-\infty}^{\infty} x(\alpha) \cdot Bm(\alpha) w_F(\alpha) d\alpha = B\int_{-\infty}^{\infty} a(\alpha) \cdot m^2(\alpha) w_F(\alpha) d\alpha$$

In the square of reference $m^2(\alpha)$ we can separate the constant average in time $\langle m^2 \rangle$ and the oscillating component $[m^2(\alpha) - \langle m^2 \rangle]$

$$y(t) = B\left\langle m^2 \right\rangle \int_{-\infty}^{\infty} a(\alpha) \cdot w_F(\alpha) d\alpha + B\int_{-\infty}^{\infty} a(\alpha) \cdot \left[m^2(\alpha) - \left\langle m^2 \right\rangle \right] w_F(\alpha) d\alpha$$

and since the weighting $w_F(\alpha)$ varies very slowly with respect to the oscillation (in a period it is almost constant) it averages the oscillation, so that we have

$$\int_{-\infty}^{\infty} a(\alpha) \cdot \left[m^2(\alpha) - \left\langle m^2 \right\rangle \right] w_F(\alpha) d\alpha \cong 0$$

In other words, the oscillating component is cut-off by the LPF as illustrated in slide 49

Time-domain study of the LIA response time

Therefore
$$y(t) \cong B\left\langle m^2 \right\rangle \int_{-\infty}^{\infty} a(\alpha) \cdot w_F(\alpha)\, d\alpha$$

the output is simply the Base-Band signal weighted by the LPF.

Since the LPF is a constant-parameter filter, we can use its δ-response *h(t)*

$$w_F(\alpha) = h(t-\alpha)$$

and confirm that the output waveform is that of the Base-Band signal filtered by the LPF

$$y(t) \cong B\left\langle m^2 \right\rangle \int_{-\infty}^{\infty} a(\alpha) \cdot h(t-\alpha)\, d\alpha = B\left\langle m^2 \right\rangle \cdot (a * h)$$

In the case of sinusoidal modulation and sinusoidal de-modulation by LIA

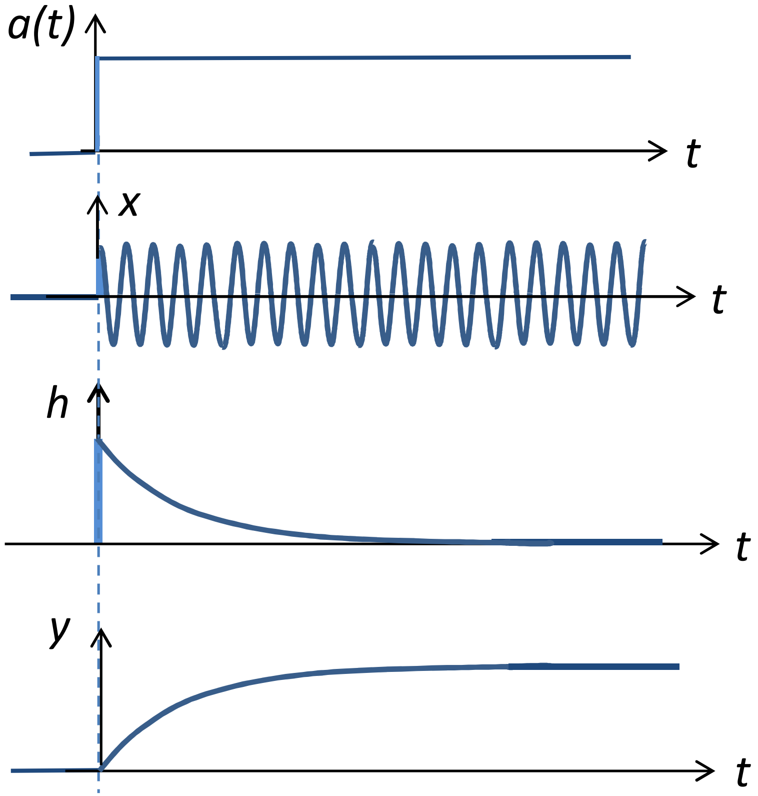
$$m(t) = \cos(\omega_m t) \qquad \left\langle m^2 \right\rangle = \frac{1}{2} \qquad \left[m^2(\alpha) - \left\langle m^2 \right\rangle \right] = \frac{1}{2}\cos(2\omega_m t)$$

and we have

$$y(t) \cong \frac{B}{2} \int_{-\infty}^{\infty} a(\alpha) \cdot h(t-\alpha)\, d\alpha = \frac{B}{2}\, a * h$$

Time-domain study of the LIA response time

Example: step strain is applied to the sensor at t = 0 and then maintained
LPF is a single pole integrator with $RC=T_F$



Base-Band signal $a(t) = 1(t)$

Modulated signal $x(t)$
(LIA input)

δ-response of LPF

$$h(t)=1(t)\cdot\exp(-t/T_F)$$

LIA output

$$y(t)\cong\frac{B}{2}1(t)\left[1-\exp(-t/T_F)\right]$$